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A generalized finite type condition for iterated function systems ☆

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Abstract

We study *iterated function systems* (IFSs) of contractive similitudes on $\mathbb{R}^d$ with overlaps. We introduce a generalized finite type condition which extends a more restrictive condition in [S.-M. Ngai, Y. Wang, Hausdorff dimension of self-similar sets with overlaps, J. London Math. Soc. (2) 63 (3) (2001) 655–672] and allows us to include some IFSs of contractive similitudes whose contraction ratios are not exponentially commensurable. We show that the generalized finite type condition implies the weak separation property. Under this condition, we can identify the attractor of the IFS with that of a graph-directed IFS, and by modifying a setup of Mauldin and Williams [R.D. Mauldin, S.C. Williams, Hausdorff dimension in graph directed constructions, Trans. Amer. Math. Soc. 309 (1988) 811–829], we can compute the Hausdorff dimension of the attractor in terms of the spectral radius of certain weighted incidence matrix.

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1. Introduction

A central problem in the theory of iterated function systems is to compute the Hausdorff dimension of the attractor. Let $\{S_i\}_{i=1}^N$ be an iterated function system of contractive similitudes on $\mathbb{R}^d$ defined as

$$S_i(x) = \rho_i R_i x + b_i, \quad i = 1, \ldots, N, \tag{1.1}$$

where $0 < \rho_i < 1$ is the contraction ratio, R_i is an orthogonal transformation, and $b_i \in \mathbb{R}^d$. Let F denote the *self-similar set* (or *attractor*) defined by the IFS, i.e., F is the unique nonempty compact set satisfying

$$F = \bigcup_{i=1}^N S_i(F).$$

It is a classical result (see Moran [19], Hutchinson [7], Falconer [5]) that if the IFS satisfies the *open set condition*, i.e., there exists a nonempty bounded open set $O \subseteq \mathbb{R}^d$ such that $\bigcup_{i=1}^N S_i(O) \subseteq O$ and $S_i(O) \cap S_j(O) = \emptyset$ for all $i \neq j$, then the Hausdorff dimension of F is the unique solution α of the equation

$$\sum_{i=1}^N \rho_i^\alpha = 1. \tag{1.2}$$

In the absence of the open set condition, much less is known. In [12], the authors introduced a weaker separation condition, known as the *weak separation property* (*WSP*) (see definition in Section 3) and studied the multifractal formalism for the associated self-similar measures under such a condition (see [12,23]). The WSP is strictly weaker than the open set condition. It is satisfied by the IFSs defining the classical Bernoulli convolutions associated with Pisot numbers (see [11–14]), and by the IFSs defining the well-known two-scale dilation equations in wavelet theory [3,12]). It is also satisfied by IFSs of the form $S_i(x) = A^{-1}(x + d_i)$, $i = 1, \ldots, N$, where $d_i \in \mathbb{Z}^d$ and $A \in M_d(\mathbb{Z})$ is an expanding (i.e., all eigenvalues are in modulus >1) integral similarity matrix (see [6,9,17]).

It is not clear how the Hausdorff dimension of the attractor can be computed by assuming the WSP alone. He et al. [6] considered IFSs with overlaps of the form $S_i(x) = A^{-1}(x + d_i)$, $i = 1, \ldots, N$, where A is an integral expanding similarity matrix and $d_i \in \mathbb{Z}^d$. By using an auxiliary tiling IFS, together with a graph-directed system, they obtained an algorithm to calculate the Hausdorff dimension of the attractor F. If F has a nonempty interior, the algorithm yields the Hausdorff dimension of the boundary of F. In another direction, by extending a method of Lalley [10] and Rao and Wen [22], Ngai and Wang [20] formulated the *finite type condition* and described a method for computing the Hausdorff dimension of the attractor in terms of the spectral radius of certain weighted incidence matrix. The finite type condition is satisfied by the three classes of IFSs satisfying the WSP that are mentioned in the previous paragraph. Nguyen [21] proved that the finite type condition implies the WSP. The finite type condition has also been extended to graph-directed IFSs by Das and Ngai [2], and has been extended by the authors to compute the Hausdorff dimension of the boundary of the attractor [15].

Although the finite type condition enlarges the class of self-similar sets for which the Hausdorff dimension can be computed, it has two shortcomings. First, it is only satisfied by IFSs of contractive similitudes with exponentially commensurable contraction ratios (see Remark 2.7). Second, it does not extend the open set condition. Recently, Lau and Wang [16] studied the following family of IFSs with overlaps. They showed that each IFS in this family has the WSP and computed the Hausdorff dimension of the attractor:

$$S_1(x)=\rho x, \qquad S_2(x)=rx+\rho(1-r), \qquad S_3(x)=rx+(1-r), \tag{1.3}$$

where $0<\rho<1$, $0<r<1$, and $\rho+2r-\rho r\leqslant 1$. This is the first example in which the WSP holds but the similitudes do not have exponentially commensurable contraction ratios and thus does not satisfy the finite type condition in [20]. In [16] the Hausdorff dimension of the attractor F is computed by expressing F as the attractor of a *countably infinite* IFS without overlaps. This approach requires a detailed analysis of the overlaps when the similitudes are iterated. Similar technical analysis is needed in showing that the IFS has the WSP. This interesting family is another main motivation for our study. We will show that both the WSP and the Hausdorff dimension of the attractor come quite easily as consequences of more general results.

Our goal in this paper is to formulate a more general finite type condition that extends both the open set condition and the finite type condition. Moreover, it does not require the similitudes in the IFS to have exponentially commensurable contraction ratios as in the original finite type condition, and thus it can include IFSs such as those in (1.3). Under such a *generalized finite type condition*, we can compute the Hausdorff dimension of the attractor by using a matrix method that can be easily applied to any of such IFSs.

The exact definition of the generalized finite type condition will be given in Section 2. The main idea is to define a suitable equivalence relation on the set of all iterates of the similitudes in the IFS and partition the iterates into equivalence classes. The generalized finite type condition holds if the number of equivalence classes is finite. In the original finite type condition, two iterates can be equivalent only if they have exponentially commensurable contraction ratios. This requirement is relaxed in the generalized finite type condition, by the introduction of more general *sequences of nested index sets* (see definition in Section 2). Our first main result is

Theorem 1.1. *Let $\{S_i\}_{i=1}^N$ be an iterated function system of contractive similitudes on $\mathbb{R}^d$. If $\{S_i\}_{i=1}^N$ is of generalized finite type, then it has the weak separation property.*

We will use the following notation throughout this paper. For any subset $E\subseteq\mathbb{R}^d$, $\dim_{\mathrm{B}}(E)$ and $\dim_{\mathrm{H}}(E)$ denote the *box dimension* and *Hausdorff dimension* of E, respectively. For $\alpha\geqslant 0$, $\mathcal{H}^\alpha(E)$ denotes the *α-dimensional Hausdorff measure* of E. The reader is referred to [5] for these definitions. We denote the *diameter* of E by $|E|$. If A is any finite or countable set, we denote by $\#A$ the *cardinality* of A. For any real number r, $[r]$ denotes the largest integer not exceeding r.

The next objective of this paper is to describe a method for computing the Hausdorff dimension of the attractor F, under the generalized finite type condition. In Section 4, we define, for each $\alpha\geqslant 0$, a finite *weighted incidence matrix* A_α. We show that the Hausdorff dimension of F is given by the unique α such that the spectral radius of A_α is equal to 1. The following is the main theorem.

Theorem 1.2. *Assume that an IFS $\{S_i\}_{i=1}^N$ of contractive similitudes on $\mathbb{R}^d$ with attractor F is of generalized finite type, and let λ_α be the spectral radius of the associated weighted incidence matrix A_α. Then*

$$\dim_{\mathrm{B}}(F) = \dim_{\mathrm{H}}(F) = \alpha,$$

where α is the unique number such that $\lambda_\alpha = 1$. Moreover, $0 < \mathcal{H}^\alpha(F) < \infty$.

The weighted incidence matrix A_α differs from the one in the original finite type condition in that its entries are, in general, functions of α instead of constants. Thus the proof for the analogous formula in [20] cannot be directly applied. Since the matrix A_α is analogous to the weighted incidence matrix in a graph-directed construction of Mauldin and Williams, we can use some ideas in [18] to obtain the upper and lower bound estimations for the Hausdorff dimension of F. In obtaining the lower bound for the Hausdorff dimension, we use a similar method as in [20] to construct a measure supported on F and then apply the mass distribution principle (see Section 4).

After this work was completed, we learned that Jin and Yau [8] have recently formulated, independently, a *general finite type condition* similar to our generalized finite type condition. They obtained some interesting results including an analogue of Theorem 1.2. The relation between the generalized finite type condition and the WSP is not studied in their paper.

This paper is organized as follows. In Section 2, we define the generalized finite type condition and give some examples. We describe two infinite graphs $\mathcal{G}$ and $\mathcal{G}_R$, which play important roles both in the definition of the generalized finite type condition and in computing the dimension of the attractor. In Section 3 we prove that the generalized finite type condition implies the weak separation property. Section 4 is devoted to the proof of Theorem 1.2. Lastly, in Section 5, we illustrate Theorem 1.2 by some examples.

2. Definition and examples of the generalized finite type condition

Let $\{S_i\}_{i=1}^N$ be an IFS of contractive similitudes on $\mathbb{R}^d$ as defined in (1.1). Define the following sets of finite indices

$$\Sigma_k := \{1, \dots, N\}^k, \quad k \geqslant 1, \qquad \text{and} \qquad \Sigma_* := \bigcup_{k \geqslant 0} \Sigma_k$$

(with $\Sigma_0 := \{\emptyset\}$). For $\boldsymbol{i} = (i_1, \dots, i_k) \in \Sigma_k$ we use the standard notation

$$S_{\boldsymbol{i}} := S_{i_1} \circ \cdots \circ S_{i_k}, \qquad \rho_{\boldsymbol{i}} := \rho_{i_1} \cdots \rho_{i_k}, \qquad R_{\boldsymbol{i}} := R_{i_1} \circ \cdots \circ R_{i_k},$$

with $\rho_\emptyset = 1$ and $S_\emptyset = R_\emptyset := I$, the identity map on $\mathbb{R}^d$. For two indices $\boldsymbol{i}, \boldsymbol{j} \in \Sigma_*$, we write $\boldsymbol{i} \preccurlyeq \boldsymbol{j}$ if $\boldsymbol{i}$ is an initial segment of $\boldsymbol{j}$ (including $\boldsymbol{i} = \boldsymbol{j}$), and write $\boldsymbol{i} \not\preccurlyeq \boldsymbol{j}$ if $\boldsymbol{i}$ is not an initial segment of $\boldsymbol{j}$. Let $|\boldsymbol{i}|$ denote the *length* of $\boldsymbol{i}$.

Consider a sequence of index sets $\{\mathcal{M}_k\}_{k=0}^\infty$, where $\mathcal{M}_k \subseteq \Sigma_*$ for all $k \geqslant 0$. Let

$$\underline{m}_k = \underline{m}_k(\mathcal{M}_k) := \min\{|\boldsymbol{i}| \colon \boldsymbol{i} \in \mathcal{M}_k\} \quad \text{and} \quad \overline{m}_k = \overline{m}_k(\mathcal{M}_k) := \max\{|\boldsymbol{i}| \colon \boldsymbol{i} \in \mathcal{M}_k\}.$$

We say that $\{\mathcal{M}_k\}_{k=0}^\infty$ is a *sequence of nested index sets* if it satisfies the following conditions:

(1) both $\{\underline{m}_k\}$ and $\{\overline{m}_k\}$ are nondecreasing, and $\lim_{k\to\infty}\underline{m}_k = \lim_{k\to\infty}\overline{m}_k = \infty$;
(2) for each $k \geqslant 0$ and all $\boldsymbol{i}, \boldsymbol{j} \in \mathcal{M}_k$, if $\boldsymbol{i} \neq \boldsymbol{j}$ then $\boldsymbol{i} \not\preccurlyeq \boldsymbol{j}$ and $\boldsymbol{j} \not\preccurlyeq \boldsymbol{i}$;
(3) for each $\boldsymbol{j} \in \Sigma_*$ with $|\boldsymbol{j}| > \overline{m}_k$, there exists $\boldsymbol{i} \in \mathcal{M}_k$ such that $\boldsymbol{i} \preccurlyeq \boldsymbol{j}$;
(4) for each $\boldsymbol{j} \in \Sigma_*$ with $|\boldsymbol{j}| < \underline{m}_k$, there exists $\boldsymbol{i} \in \mathcal{M}_k$ such that $\boldsymbol{j} \preccurlyeq \boldsymbol{i}$;
(5) there exists a positive integer L, independent of k, such that for all $\boldsymbol{i} \in \mathcal{M}_k$ and $\boldsymbol{j} \in \mathcal{M}_{k+1}$ with $\boldsymbol{i} \preccurlyeq \boldsymbol{j}$, we have $|\boldsymbol{j}| - |\boldsymbol{i}| \leqslant L$.

(We allow $\mathcal{M}_k \cap \mathcal{M}_{k+1} \neq \emptyset$. Very often, $\bigcup_{k=0}^{\infty}\mathcal{M}_k$ is a proper subset of Σ_*.) Note that if $\Omega \subseteq \mathbb{R}^d$ is nonempty, bounded, and *invariant* under $\{S_i\}_{i=1}^N$, i.e., $\bigcup_{i=1}^N S_i(\Omega) \subseteq \Omega$, then $\{\bigcup_{\boldsymbol{i}\in\mathcal{M}_k} S_{\boldsymbol{i}}(\Omega)\}_{k=0}^{\infty}$ is a sequence of nested subsets of $\mathbb{R}^d$. The sequences of nested index sets generalize the notion of "level of iteration." We mention two examples below, and construct a different one in Appendix A. Other sequences of nested index sets can be constructed easily.

Example 2.1. Let $\mathcal{M}_k = \Sigma_k$ for all $k \geqslant 0$. It is easy to see that conditions (1)–(5) are satisfied.

The sequence of nested index sets in Example 2.1 is the most standard one and is used in the case the linear parts of the similitudes, $\rho_i R_i$, $i = 1, \dots, N$, are all equal. It is also used to handle iterations of similitudes whose contractions ratios are not exponentially commensurable (see Example 2.8).

The following is the sequence of nested index sets used in the original finite type condition [20].

Example 2.2. For $k \geqslant 0$, let

$$\mathcal{M}_k = \Lambda_k := \bigl\{\boldsymbol{j} = (j_1, \dots, j_n) \in \Sigma_*\colon\ \rho_{\boldsymbol{j}} \leqslant \rho^k < \rho_{j_1\dots j_{n-1}}\bigr\}.$$

It is easy to see that conditions (1)–(4) hold. Condition (5) holds by taking $L = [\ln\rho/\ln\rho_{\max}] + 1$, where $\rho := \min\{\rho_i\colon\ 1 \leqslant i \leqslant N\}$ and $\rho_{\max} := \max\{\rho_i\colon\ 1 \leqslant i \leqslant N\}$.

Fix a sequence of nested index sets $\{\mathcal{M}_k\}_{k=0}^{\infty}$. For each integer $k \geqslant 0$, let $\mathcal{V}_k$ be the set of *vertices* (with respect to $\{\mathcal{M}_k\}_{k=0}^{\infty}$) defined as

$$\mathcal{V}_0 := \bigl\{(I, 0)\bigr\} \quad \text{and} \quad \mathcal{V}_k := \bigl\{(S_{\boldsymbol{i}}, k)\colon\ \boldsymbol{i} \in \mathcal{M}_k\bigr\} \quad \text{for all } k \geqslant 1.$$

We call $(I, 0)$ the *root vertex* and denote it by $\boldsymbol{v}_{\text{root}}$. Let $\mathcal{V} := \bigcup_{k\geqslant 0}\mathcal{V}_k$.

For $\boldsymbol{v} = (S_{\boldsymbol{i}}, k) \in \mathcal{V}_k$, we introduce the convenient notation $S_{\boldsymbol{v}} := S_{\boldsymbol{i}}$ and $\rho_{\boldsymbol{v}} := \rho_{\boldsymbol{i}}$. Note that it is possible that $\boldsymbol{v} = (S_{\boldsymbol{i}}, k) = (S_{\boldsymbol{j}}, k)$ with $\boldsymbol{i} \neq \boldsymbol{j}$. The notation $S_{\boldsymbol{v}}$ allows us to refer to a vertex in $\mathcal{V}_k$ without explicitly specifying the index $\boldsymbol{i}$.

Fix any nonempty bounded open set Ω which is invariant under $\{S_i\}_{i=1}^N$. Two vertices $\boldsymbol{v}, \boldsymbol{v}' \in \mathcal{V}_k$ (allowing $\boldsymbol{v} = \boldsymbol{v}'$) are *neighbors* (with respect to Ω) if $S_{\boldsymbol{v}}(\Omega) \cap S_{\boldsymbol{v}'}(\Omega) \neq \emptyset$. The set of vertices

$$\Omega(\boldsymbol{v}) := \{\boldsymbol{v}'\colon\ \boldsymbol{v}' \text{ is a neighbor of } \boldsymbol{v}\}$$

is called the *neighborhood* of $\boldsymbol{v}$ (with respect to Ω). Note that $\boldsymbol{v} \in \Omega(\boldsymbol{v})$ by definition.

We define an equivalence relation on $\mathcal{V}$ to identify neighborhoods that are isomorphic in the sense that they behave the same upon iteration.

Definition 2.1. Two vertices $\boldsymbol{v} \in \mathcal{V}_k$ and $\boldsymbol{v}' \in \mathcal{V}_{k'}$ are *equivalent*, denoted by $\boldsymbol{v} \sim_\Omega \boldsymbol{v}'$ (or simply $\boldsymbol{v} \sim \boldsymbol{v}'$), if, for $\tau := S_{\boldsymbol{v}'} \circ S_{\boldsymbol{v}}^{-1} : \mathbb{R}^d \to \mathbb{R}^d$, the following conditions are satisfied:

(i) $\{S_{\boldsymbol{u}'}: \boldsymbol{u}' \in \Omega(\boldsymbol{v}')\} = \{\tau \circ S_{\boldsymbol{u}}: \boldsymbol{u} \in \Omega(\boldsymbol{v})\}$;
(ii) for $\boldsymbol{u} \in \Omega(\boldsymbol{v})$ and $\boldsymbol{u}' \in \Omega(\boldsymbol{v}')$ such that $S_{\boldsymbol{u}'} = \tau \circ S_{\boldsymbol{u}}$, and for any positive integer $\ell \geqslant 1$, an index $\boldsymbol{i} \in \Sigma_*$ satisfies $(S_{\boldsymbol{u}} \circ S_{\boldsymbol{i}}, k+\ell) \in \mathcal{V}_{k+\ell}$ if and only if it satisfies $(S_{\boldsymbol{u}'} \circ S_{\boldsymbol{i}}, k'+\ell) \in \mathcal{V}_{k'+\ell}$.

It is easy to see that $\sim$ is an equivalence relation. We denote the equivalence class containing $\boldsymbol{v}$ by $[\boldsymbol{v}]$ (or $[\boldsymbol{v}]_\Omega$) and call it the *neighborhood type* of $\boldsymbol{v}$ (with respect to Ω). We remark that (ii) says roughly that two vertices of the same neighborhood type have equivalent offspring. We will prove this rigorously in Proposition 2.4.

We define two important infinite directed graphs $\mathcal{G}$ and $\mathcal{G}_R$. The graph $\mathcal{G}$ has vertex set $\mathcal{V}$ and directed edges defined as follows. Let $\boldsymbol{v} \in \mathcal{V}_k$ and $\boldsymbol{u} \in \mathcal{V}_{k+1}$. Suppose there exist $\boldsymbol{i} \in \mathcal{M}_k$, $\boldsymbol{j} \in \mathcal{M}_{k+1}$, and $\boldsymbol{l} \in \Sigma_*$ such that

$$\boldsymbol{v} = (S_{\boldsymbol{i}}, k), \qquad \boldsymbol{u} = (S_{\boldsymbol{j}}, k+1), \quad \text{and} \quad \boldsymbol{j} = (\boldsymbol{i}, \boldsymbol{l}).$$

Then we connect a directed edge $\boldsymbol{l}$ from $\boldsymbol{v}$ to $\boldsymbol{u}$ and denote this by $\boldsymbol{v} \xrightarrow{\boldsymbol{l}} \boldsymbol{u}$. We call $\boldsymbol{v}$ a *parent* of $\boldsymbol{u}$ in $\mathcal{G}$ and $\boldsymbol{u}$ an *offspring* (or *descendant*) of $\boldsymbol{v}$ in $\mathcal{G}$. We write $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{E}$ is the set of all directed edges defined above.

The *reduced graph* $\mathcal{G}_R$ is obtained from $\mathcal{G}$ by first removing all but the smallest (in the lexicographical order) directed edge going to a vertex. More precisely, let $\boldsymbol{v}_k \xrightarrow{\boldsymbol{l}_k} \boldsymbol{u}$, $k = 1, \dots, m$, be all the directed edges going to the vertex $\boldsymbol{u} \in \mathcal{V}_{k+1}$, where $\boldsymbol{v}_k \in \mathcal{V}_k$ are distinct and thus the $\boldsymbol{l}_k$ are also distinct. Suppose $\boldsymbol{l}_1 < \cdots < \boldsymbol{l}_m$ in the lexicographical order (or any fixed order). Then we keep only $\boldsymbol{l}_1$ in the reduced graph and remove all the edges $\boldsymbol{l}_k$, $2 \leqslant k \leqslant m$ (see Example 2.8).

Next, we notice that if $\mathcal{M}_k = \Sigma_k$ for all $k \geqslant 0$, then each vertex in $\mathcal{V}$ has an offspring in $\mathcal{G}_R$ which is connected by the edge $\boldsymbol{l} = 1$. However, this is not necessarily the case for other sequences of nested index sets. It is possible that a vertex in $\mathcal{V}$ does not have any offspring in $\mathcal{G}_R$. We will provide a concrete example in Appendix A.

To finish the construction of the reduced graph, we remove all vertices that do not have offspring in $\mathcal{G}_R$, together with all the vertices and edges leading only to them. We denote the resulting graph by the same symbol $\mathcal{G}_R$ and write $\mathcal{G}_R = (\mathcal{V}_R, \mathcal{E}_R)$, where $\mathcal{V}_R$ is the set of all vertices and $\mathcal{E}_R$ is the set of all edges.

Remark 2.3. It follows from the invariance of Ω under $\{S_i\}_{i=1}^N$ that only vertices in $\Omega(\boldsymbol{v})$ can be parents of any offspring of $\boldsymbol{v}$ in $\mathcal{G}$. In fact, if $\boldsymbol{u} = (S_{\boldsymbol{v}} \circ S_{\boldsymbol{l}}, k+1) \in \mathcal{V}_{k+1}$ is an offspring of $\boldsymbol{v}$ in $\mathcal{G}$ and if $\boldsymbol{w} \in \mathcal{V}_k \setminus \Omega(\boldsymbol{v})$, then for any index $\boldsymbol{i} \in \Sigma_*$,

$$S_{\boldsymbol{w}} \circ S_{\boldsymbol{i}}(\Omega) \cap S_{\boldsymbol{u}}(\Omega) \subseteq S_{\boldsymbol{w}}(\Omega) \cap S_{\boldsymbol{v}}(\Omega) = \emptyset.$$

Hence $\boldsymbol{w}$ cannot be a parent of $\boldsymbol{u}$.

Proposition 2.4. *Let $\boldsymbol{v} \in \mathcal{V}_k$ and $\boldsymbol{v}' \in \mathcal{V}_{k'}$ be two vertices with offspring $\boldsymbol{u}_1, \dots, \boldsymbol{u}_m$ and $\boldsymbol{u}'_1, \dots, \boldsymbol{u}'_\ell$ in $\mathcal{G}_R$, respectively. Suppose $[\boldsymbol{v}] = [\boldsymbol{v}']$ and let*

$$\Omega(\boldsymbol{v}) = \{\boldsymbol{v}_0 = \boldsymbol{v}, \boldsymbol{v}_1, \dots, \boldsymbol{v}_n\} \quad \text{and} \quad \Omega(\boldsymbol{v}') = \{\boldsymbol{v}'_0 = \boldsymbol{v}', \boldsymbol{v}'_1, \dots, \boldsymbol{v}'_n\}$$

such that $S_{\boldsymbol{v}'_j} = \tau \circ S_{\boldsymbol{v}_j}$ *for* $0 \leqslant j \leqslant n$, *where* $\tau = S_{\boldsymbol{v}'} \circ S_{\boldsymbol{v}}^{-1}$. *Then the following statements hold.*

(a) *Let* $0 \leqslant i, j \leqslant n$ *and suppose, by the definition of* $\sim$, *that* $\boldsymbol{i}_1, \boldsymbol{i}_2 \in \mathcal{E}$ *such that*

$$\boldsymbol{v}_i \xrightarrow{\boldsymbol{i}_1} \boldsymbol{u}, \qquad \boldsymbol{v}_j \xrightarrow{\boldsymbol{i}_2} \boldsymbol{w},$$
$$\boldsymbol{v}'_i \xrightarrow{\boldsymbol{i}_1} \boldsymbol{u}', \qquad \boldsymbol{v}'_j \xrightarrow{\boldsymbol{i}_2} \boldsymbol{w}'.$$

Then $\boldsymbol{u} = \boldsymbol{w}$ *if and only if* $\boldsymbol{u}' = \boldsymbol{w}'$, *and* $\boldsymbol{u}, \boldsymbol{w}$ *are neighbors if and only if* $\boldsymbol{u}', \boldsymbol{w}'$ *are.*

(b) *The following equality holds* (*counting multiplicity*):

$$\{[\boldsymbol{u}_i]\colon\ 1 \leqslant i \leqslant m\} = \{[\boldsymbol{u}'_i]\colon\ 1 \leqslant i \leqslant \ell\}. \tag{2.1}$$

In particular, $m = \ell$.

Proof. (a) We notice that

$$S_{\boldsymbol{u}'} = S_{\boldsymbol{v}'_i} \circ S_{\boldsymbol{i}_1} = \tau \circ S_{\boldsymbol{v}_i} \circ S_{\boldsymbol{i}_1} = \tau \circ S_{\boldsymbol{u}}. \tag{2.2}$$

Similarly, $S_{\boldsymbol{w}'} = \tau \circ S_{\boldsymbol{w}}$. Hence $S_{\boldsymbol{u}} = S_{\boldsymbol{w}}$ if and only if $S_{\boldsymbol{u}'} = S_{\boldsymbol{w}'}$. That is, $\boldsymbol{u} = \boldsymbol{w}$ if and only if $\boldsymbol{u}' = \boldsymbol{w}'$. The second part follows from the following equivalences:

$$S_{\boldsymbol{u}}(\Omega) \cap S_{\boldsymbol{w}}(\Omega) \neq \emptyset \quad \Leftrightarrow \quad \tau \circ S_{\boldsymbol{u}}(\Omega) \cap \tau \circ S_{\boldsymbol{w}}(\Omega) \neq \emptyset \quad \Leftrightarrow \quad S_{\boldsymbol{u}'}(\Omega) \cap S_{\boldsymbol{w}'}(\Omega) \neq \emptyset.$$

This proves (a).

(b) We let $\mathcal{U}$ and $\mathcal{U}'$ be the sets of offspring of the vertices in $\Omega(\boldsymbol{v})$ and $\Omega(\boldsymbol{v}')$, respectively. Define a map $\tilde{\tau}: \mathcal{U} \to \mathcal{U}'$ as follows. Suppose $\boldsymbol{u}$ is an offspring of $\boldsymbol{v}_j$ in $\mathcal{G}$ by an edge $\boldsymbol{i}$. Then we let $\tilde{\tau}(\boldsymbol{u})$ be the offspring of $\boldsymbol{v}'_j$ by the edge $\boldsymbol{i}$. The definition of $\sim$ and part (a) above imply that $\tilde{\tau}$ is well defined and bijective. Moreover, by (2.2) we have

$$S_{\tilde{\tau}(\boldsymbol{u})} = \tau \circ S_{\boldsymbol{u}}. \tag{2.3}$$

By (a), $\boldsymbol{u}$ is an offspring of $\boldsymbol{v}$ in $\mathcal{G}_R$ if and only if $\tilde{\tau}(\boldsymbol{u})$ is an offspring of $\boldsymbol{v}'$ in $\mathcal{G}_R$. Therefore $m = \ell$. Now, combining Remark 2.3, equality (2.3) and part (a) yields $[\tilde{\tau}(\boldsymbol{u}_i)] = [\boldsymbol{u}_i]$ for $1 \leqslant i \leqslant m$, and thus (2.1) follows. This completes the proof. □

Definition 2.2. We say that an IFS of contractive similitudes on $\mathbb{R}^d$ defined as in (1.1) is of *generalized finite type*, or that it satisfies the *generalized finite type condition*, if there exists a nonempty bounded invariant open set Ω such that, with respect to some sequence of nested index sets $\{\mathcal{M}_k\}_{k=0}^{\infty}$, $\mathcal{V}/_{\sim} = \{[\boldsymbol{v}]_{\Omega}\colon\ \boldsymbol{v} \in \mathcal{V}\}$ is a finite set. We call such an Ω a *basic set for the generalized finite type condition*, or just a *basic set*.

Suppose there exists some $k \geqslant 1$ such that none of the vertices in $\mathcal{V}_k$ are of a new neighborhood type. Then Proposition 2.4 implies that the IFS is of generalized finite type. We illustrate this by some examples.

Example 2.5. If $\{S_i\}_{i=1}^{N}$ satisfies the open set condition (see [5,7]), then it is of generalized finite type.

Proof. Let Ω be an open set condition set and for each $k \geqslant 0$ let $\mathcal{M}_k = \Sigma_k$. For each $\boldsymbol{v} \in \mathcal{V}_1 = \{(S_i, 1)\colon 1 \leqslant i \leqslant N\}$, the open set condition implies that $\Omega(\boldsymbol{v}) = \{\boldsymbol{v}\}$. Let $\tau = I \circ S_{\boldsymbol{v}}^{-1}$. Then $\tau \circ S_{\boldsymbol{v}} = I$ and it follows that $\boldsymbol{v} \sim (I, 0) = \boldsymbol{v}_{\text{root}}$. Proposition 2.4 now implies that $\mathcal{V}/_\sim = \{[\boldsymbol{v}_{\text{root}}]\}$ and the result follows. □

Example 2.6. If $\{S_i\}_{i=1}^N$ is of finite type [20] then it is of generalized finite type.

Proof. Let Ω be a nonempty bounded invariant open set. Let $\rho := \min\{\rho_i\colon 1 \leqslant i \leqslant N\}$ and for $k \geqslant 0$ define

$$\mathcal{M}_k = \Lambda_k = \big\{\boldsymbol{j} = (j_1, \ldots, j_n) \in \Sigma_*\colon \rho_{\boldsymbol{j}} \leqslant \rho^k < \rho_{j_1 \ldots j_{n-1}}\big\}.$$

The definitions of $\mathcal{V}_k$, $\mathcal{V}$ and $\Omega(\boldsymbol{v})$ now coincide with their original definitions in [20]. We recall that in the original definition, two vertices $\boldsymbol{v}, \boldsymbol{v}'$ are *equivalent* if there exists a similitude of the form $\tau(x) = \rho^{k'-k} Ux + c$, where U is orthogonal and $c \in \mathbb{R}^d$ such that $S_{\boldsymbol{v}'} = \tau \circ S_{\boldsymbol{v}}$ and $\{S_{\boldsymbol{u}'}\colon \boldsymbol{u}' \in \Omega(\boldsymbol{v}')\} = \{\tau \circ S_{\boldsymbol{u}}\colon \boldsymbol{u} \in \Omega(\boldsymbol{v})\}$. Thus, the original equivalence relation $\sim$ satisfies condition (i) in the present definition. Recall also that the IFS is said to be of *finite type* if there exists a nonempty bounded invariant open set Ω (called a *finite type condition set*) with respect to which $\mathcal{V}/_\sim$ is a finite set.

We need to check that the original definition of $\sim$ also satisfies condition (ii) in the present definition. Let $\boldsymbol{i} = (i_1, \ldots, i_t)$ and assume $(S_{\boldsymbol{u}} \circ S_{\boldsymbol{i}}, k + \ell) \in \mathcal{V}_{k+\ell}$ for some $\ell \geqslant 1$. Then

$$\rho_{\boldsymbol{u}} \rho_{\boldsymbol{i}} \leqslant \rho^{k+\ell} < \rho_{\boldsymbol{u}} \rho_{i_1 \ldots i_{t-1}}.$$

Since $S_{\boldsymbol{u}'} \circ S_{\boldsymbol{i}} = \tau \circ S_{\boldsymbol{u}} \circ S_{\boldsymbol{i}}$, we have

$$\rho_{\boldsymbol{u}'} \rho_{\boldsymbol{i}} = \rho^{k'-k} \rho_{\boldsymbol{u}} \rho_{\boldsymbol{i}} \leqslant \rho^{k'-k} \rho^{k+\ell} = \rho^{k'+\ell}.$$

On the other hand,

$$\rho_{\boldsymbol{u}'} \rho_{i_1 \ldots i_{t-1}} = \rho^{k'-k} \rho_{\boldsymbol{u}} \rho_{i_1 \ldots i_{t-1}} > \rho^{k'-k} \rho^{k+\ell} = \rho^{k'+\ell}.$$

Thus, $(S_{\boldsymbol{u}'} \circ S_{\boldsymbol{i}}, k' + \ell) \in \mathcal{V}_{k'+\ell}$. The same proof shows that if $\boldsymbol{i} \in \Sigma_*$ and $(S_{\boldsymbol{u}'} \circ S_{\boldsymbol{i}}, k' + \ell) \in \mathcal{V}_{k'+\ell}$ for some $\ell > 0$, then $(S_{\boldsymbol{u}} \circ S_{\boldsymbol{i}}, k + \ell) \in \mathcal{V}_{k+\ell}$. Thus condition (ii) holds.

Now let Ω be a finite type condition set. By the finite type condition $\mathcal{V}/_\sim$ is finite and thus the IFS is of generalized finite type. □

Remark 2.7. The condition $\tau(x) = \rho^{k'-k} Ux + c$ in the definition of the finite type condition in [20] forces $\{\rho_i\}_{i=1}^N$ to be exponentially commensurable.

Proof. Let $\boldsymbol{v} \in \mathcal{V}_k$ and $\boldsymbol{v}' \in \mathcal{V}_{k'}$. Then

$$\boldsymbol{v}' \in [\boldsymbol{v}] \quad \Rightarrow \quad S_{\boldsymbol{v}'} = \tau \circ S_{\boldsymbol{v}} \quad \Rightarrow \quad \rho^{-k'} \rho_{\boldsymbol{v}'} = \rho^{-k} \rho_{\boldsymbol{v}}.$$

Thus, $\{\rho^{-k}\rho_{\boldsymbol{j}}\colon \boldsymbol{j} \in \Lambda_k,\ k \geqslant 0\}$ is finite. Fix $i = 1, \ldots, N$. For each k, there exists ℓ_k such that $(\underbrace{1, \ldots, 1}_{\ell_k}) \in \Lambda_k$. Then by finiteness, there exist k, k' with $k' \neq k$ such that

$$\rho^{-k}\rho_i^{\ell_k} = \rho^{-k'}\rho_i^{\ell_{k'}}, \quad \text{i.e.,} \quad \rho_i^{\ell_k - \ell_{k'}} = \rho^{k-k'}.$$

Hence, the ρ_i are exponentially commensurable. □

The following example from [16] is an IFS of contractive similitudes whose contraction ratios are not exponentially commensurable.

Example 2.8. Let $\{S_i\}_{i=1}^3$ be an IFS on $\mathbb{R}$ as defined in (1.3):

$$S_1(x) = \rho x, \qquad S_2(x) = rx + \rho(1-r), \qquad S_3(x) = rx + (1-r),$$

where $0 < \rho < 1$, $0 < r < 1$, and $\rho + 2r - \rho r \leqslant 1$. Then $\{S_i\}_{i=1}^3$ is of generalized finite type.

Proof. Let $\Omega = (0,1)$. For each $k \geqslant 0$ let $\mathcal{M}_k = \Sigma_k$. Upon iterating the IFS once, the root vertex generates three vertices:

$$\boldsymbol{v}_{\text{root}} = (I, 0) \quad \longrightarrow \quad \boldsymbol{v}_1 = (S_1, 1), \quad \boldsymbol{v}_2 = (S_2, 1), \quad \boldsymbol{v}_3 = (S_3, 1).$$

Since $\Omega(\boldsymbol{v}_3) = \{\boldsymbol{v}_3\}$, it follows that $\boldsymbol{v}_3 \sim \boldsymbol{v}_{\text{root}}$ with $\tau = I \circ S_{\boldsymbol{v}_3}^{-1} = S_{\boldsymbol{v}_3}^{-1}$. It is easy to check that $[\boldsymbol{v}_{\text{root}}]$, $[\boldsymbol{v}_1]$ and $[\boldsymbol{v}_2]$, denoted respectively by $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$, are distinct neighborhood types. Moreover, it follows from definitions that $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ can be expressed explicitly as

$$\begin{aligned}
\mathcal{T}_1 &= \{\boldsymbol{v} \in \mathcal{G}\colon \boldsymbol{v} \sim \boldsymbol{v}_{\text{root}}\} = \bigl\{\boldsymbol{v} \in \mathcal{G}\colon \Omega(\boldsymbol{v}) = \{\boldsymbol{v}\}\bigr\},\\
\mathcal{T}_2 &= \{\boldsymbol{v} \in \mathcal{G}\colon \boldsymbol{v} \sim \boldsymbol{v}_1\} = \bigl\{\boldsymbol{v} \in \mathcal{G}\colon \Omega(\boldsymbol{v}) = \{\boldsymbol{v}, \boldsymbol{v}'\} \text{ and } S_{\boldsymbol{v}_1} \circ S_{\boldsymbol{v}}^{-1} \circ S_{\boldsymbol{v}'} = S_{\boldsymbol{v}_2}\bigr\},\\
\mathcal{T}_3 &= \{\boldsymbol{v} \in \mathcal{G}\colon \boldsymbol{v} \sim \boldsymbol{v}_2\} = \bigl\{\boldsymbol{v} \in \mathcal{G}\colon \Omega(\boldsymbol{v}) = \{\boldsymbol{v}, \boldsymbol{v}'\} \text{ and } S_{\boldsymbol{v}_2} \circ S_{\boldsymbol{v}}^{-1} \circ S_{\boldsymbol{v}'} = S_{\boldsymbol{v}_1}\bigr\}.
\end{aligned}$$

Upon one more iteration, $\boldsymbol{v}_1$ generates three offspring in $\mathcal{G}$,

$$\boldsymbol{v}_1 \quad \longrightarrow \quad \boldsymbol{v}_4 = (S_1S_1, 2), \qquad \boldsymbol{v}_5 = (S_1S_2, 2), \qquad \boldsymbol{v}_6 = (S_1S_3, 2),$$

and $\boldsymbol{v}_2$ also generates three offspring in $\mathcal{G}$,

$$\boldsymbol{v}_2 \quad \longrightarrow \quad \boldsymbol{v}_7 = (S_2S_1, 2), \qquad \boldsymbol{v}_8 = (S_2S_2, 2), \qquad \boldsymbol{v}_9 = (S_2S_3, 2).$$

By using the above explicit expressions of $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$, it is straightforward to verify that

$$[\boldsymbol{v}_4] = [\boldsymbol{v}_6] = [\boldsymbol{v}_7] = \mathcal{T}_2, \quad [\boldsymbol{v}_5] = [\boldsymbol{v}_8] = \mathcal{T}_3 \quad \text{and} \quad [\boldsymbol{v}_9] = \mathcal{T}_1.$$

Since no new neighborhood types are generated, Proposition 2.4 now implies that $\mathcal{V}/_\sim = \{[\boldsymbol{v}_{\text{root}}], [\boldsymbol{v}_1], [\boldsymbol{v}_2]\}$ and thus the generalized finite type condition holds. □

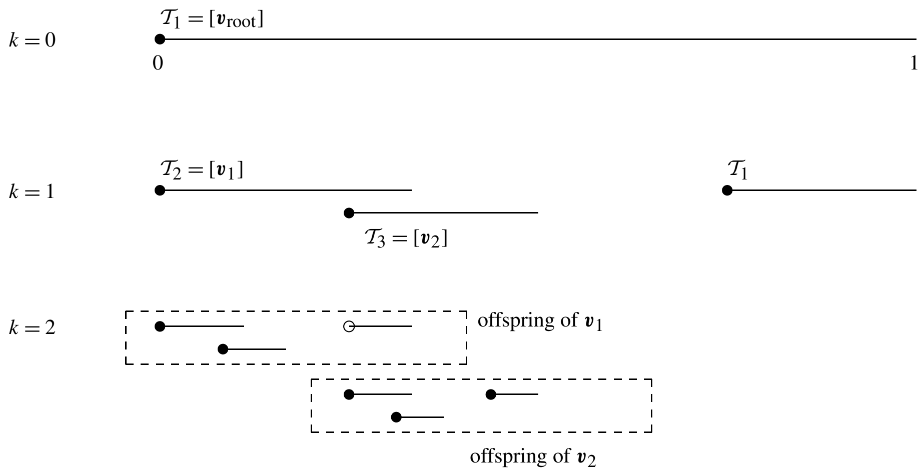


Fig. 1. Vertices in $\mathcal{V}_k$ for $k = 0, 1, 2$ in Example 2.8, drawn for $\rho = 1/3$ and $r = 1/4$.

To construct the reduced graph $\mathcal{G}_R$ for Example 2.8, we notice that $(S_1S_3)(x) = (S_2S_1)(x) = \rho rx + \rho(1-r)$. That is, the offspring $\boldsymbol{v}_6$ of $\boldsymbol{v}_1$ is identical with the offspring $\boldsymbol{v}_7$ of $\boldsymbol{v}_2$. Moreover,

$$\boldsymbol{v}_1 \xrightarrow{(3)} \boldsymbol{v}_6 \quad \text{and} \quad \boldsymbol{v}_2 \xrightarrow{(1)} \boldsymbol{v}_7.$$

Since $1 < 3$ in the lexicographical order, according to the construction $\mathcal{G}_R$, the edge (3) connecting $\boldsymbol{v}_1$ to $\boldsymbol{v}_6$ is removed. We will use this result in Example 5.1.

Figure 1 illustrates how the distinct neighborhood types are generated. It is drawn for the case $\rho = 1/3$ and $r = 1/4$. Overlapping vertices are separated vertically to show distinction. Iterates of the point 0 under the similitudes are represented by dots (or circles). For $k = 2$, only offspring of $\boldsymbol{v}_1 = (S_1, 1)$ and $\boldsymbol{v}_2 = (S_2, 1)$ are shown. The edge connecting $\boldsymbol{v}_2$ to the offspring indicated by a circle is to be removed when constructing $\mathcal{G}_R$.

The following example shows that it is possible for an IFS to be of generalized finite type with respect to one sequence of nested index sets but $\mathcal{V}/\sim$ is not finite if we choose another sequence of nested index sets.

Example 2.9. Consider the IFS

$$S_1(x) = \frac{1}{2}x, \qquad S_2(x) = \frac{1}{2}x + \frac{1}{2}, \qquad S_3(x) = \frac{1}{4}x.$$

If we let $\mathcal{M}_k = \Lambda_k$ for $k \geqslant 0$, then with respect to any nonempty bounded invariant open set Ω, $\mathcal{V}/\sim$ is finite and thus the IFS is of generalized finite type. However, if we take $\mathcal{M}_k = \Sigma_k$ for $k \geqslant 0$, then with respect to any nonempty bounded invariant open set Ω, $\mathcal{V}/\sim$ is infinite.

Proof. The IFS is of finite type with respect to any nonempty bounded invariant open set (see [20, Theorem 2.9]) and thus $\mathcal{V}/\sim$ is finite if we take $\mathcal{M}_k = \Lambda_k$, $k \geqslant 0$ (see Example 2.6).

To show that $\mathcal{V}/\sim$ is always infinite if we take $\mathcal{M}_k = \Sigma_k$ for $k \geqslant 0$, we first let $\Omega = (0,1)$. Let $\mathbf{1}_k := (1,\dots,1) \in \Sigma_k$ and $\mathbf{3}_k := (3,\dots,3) \in \Sigma_k$. Then for all $k \geqslant 1$, $(S_{\mathbf{1}_k}, k)$ and $(S_{\mathbf{3}_k}, k)$ are neighbors with

$$\frac{|S_{\mathbf{1}_k}(0,1)|}{|S_{\mathbf{3}_k}(0,1)|} = \frac{1/2^k}{1/4^k} = 2^k,$$

which tends to ∞ as $k \to \infty$. Hence for all $m > k$, $[S_{\mathbf{1}_m}] \neq [S_{\mathbf{1}_k}]$, which implies that $\mathcal{V}/\sim$ is infinite.

Since any nonempty bounded invariant open set must be of the form $\Omega = \bigcup_n (a_n, a_{n+1})$ and satisfy $\overline{\Omega} \supseteq [0,1]$, the same proof above shows that $\mathcal{V}/\sim$ is infinite with respect to any such Ω. □

3. Relationship with the weak separation property

The weak separation property was introduced by the authors to study the multifractal formalism of self-similar measures defined by IFSs with overlaps (see [12,16,23]). It is proved by Nguyen [21] that the finite type condition implies the weak separation property. In this section we strengthen this result by showing that the generalized finite type condition also implies the weak separation property. For an IFS $\{S_i\}_{i=1}^N$ of contractive similitudes on $\mathbb{R}^d$, where $S_i(x) = \rho_i R_i x + b_i$, recall that $\rho := \min\{\rho_i\colon 1 \leqslant i \leqslant N\}$ and $\rho_{\max} := \max\{\rho_i\colon 1 \leqslant i \leqslant N\}$. For $k \geqslant 0$ and $0 < b < 1$, define

$$\mathcal{I}_b := \bigl\{\boldsymbol{j} = (j_1,\dots,j_n) \in \Sigma_*\colon\ \rho_{\boldsymbol{j}} \leqslant b < \rho_{j_1\dots j_{n-1}}\bigr\},$$
$$\mathcal{A}_b := \{S_{\boldsymbol{j}}\colon\ \boldsymbol{j} \in \mathcal{I}_b\}.$$

For convenience, let us recall the definition of the weak separation property (see [12,16,23]).

Definition 3.1. An IFS $\{S_i\}_{i=1}^N$ of contractive similitudes on $\mathbb{R}^d$ has the *weak separation property* (*WSP*) if there exists some $x_0 \in \mathbb{R}^d$ and $\ell \in \mathbb{N}$ such that for any $\boldsymbol{i} \in \Sigma_*$ and $0 < b < 1$, any closed ball with radius b contains no more than ℓ distinct points of the form $S(S_{\boldsymbol{i}}(x_0))$, $S \in \mathcal{A}_b$.

The following proposition can be derived by using the proof of [23, Theorem 1]. We include a proof for completeness. Let $\mathcal{L}$ denote the d-dimensional Lebesgue measure.

Lemma 3.1. *Let $\{S_i\}_{i=1}^N$ be an IFS of contractive similitudes on $\mathbb{R}^d$. Suppose there exists $\gamma \in \mathbb{N}$ and a bounded invariant subset $\Omega \subseteq \mathbb{R}^d$ with $\mathcal{L}(\Omega) > 0$ such that for any $x \in \mathbb{R}^d$ and $0 < b < 1$, $\#\{S \in \mathcal{A}_b\colon x \in S(\Omega)\} \leqslant \gamma$. Then $\{S_i\}_{i=1}^N$ has the WSP.*

Proof. Let $x_0 \in \Omega$, $\boldsymbol{i} \in \Sigma_*$, $0 < b < 1$, and B be a closed ball of radius b. Let $S \in \mathcal{A}_b$ such that $S(S_{\boldsymbol{i}}(x_0)) \in B$. Since $S_{\boldsymbol{i}}(x_0) \subseteq S_{\boldsymbol{i}}(\Omega) \subseteq \Omega$, it follows that $S(\Omega) \cap B \neq \emptyset$. Thus $S(\Omega) \subseteq \widetilde{B}$, where $\widetilde{B}$ is concentric with B and has radius $b(1 + |\Omega|)$. By assumption, each point in $\widetilde{B}$ is covered by no more than γ of the sets $S(\Omega)$, $S \in \mathcal{A}_b$. Hence

$$(b\rho)^d \mathcal{L}(\Omega)\#\bigl\{S \in \mathcal{A}_b\colon\ S\bigl(S_{\boldsymbol{i}}(x_0)\bigr) \in B\bigr\}$$
$$\leqslant \sum\bigl\{\mathcal{L}\bigl(S(\Omega)\bigr)\colon\ S \in \mathcal{A}_b,\ S\bigl(S_{\boldsymbol{i}}(x_0)\bigr) \in B\bigr\} \leqslant \gamma \mathcal{L}(\widetilde{B}).$$

It follows that

$$\#\{S \in \mathcal{A}_b:\ S(S_{\boldsymbol{i}}(x_0)) \in B\} \leqslant \frac{\gamma c_d (1+|\Omega|)^d}{\rho^d \mathcal{L}(\Omega)} := C,$$

where c_d is the volume of the unit ball in $\mathbb{R}^d$. Notice that for any $S_1, S_2 \in \mathcal{A}_b$, $S_1(S_{\boldsymbol{i}}(x_0)) \neq S_2(S_{\boldsymbol{i}}(x_0))$ implies that $S_1 \neq S_2$. Consequently,

$$\#\{S(S_{\boldsymbol{i}}(x_0)) \in B:\ S \in \mathcal{A}_b\} \leqslant \#\{S \in \mathcal{A}_b:\ S(S_{\boldsymbol{i}}(x_0)) \in B\} = C,$$

completing the proof. □

We remark that in Lemma 3.1 the assumption that Ω is invariant under $\{S_i\}_{i=1}^N$ can be dropped. The same conclusion can be derived from the proof of [23, Theorem 1.1] but the derivation is more complicated. For simplicity and for our purposes, it is sufficient to include this assumption in the lemma.

Proof of Theorem 1.1. Fix a sequence of nested index sets $\{\mathcal{M}_k\}_{k=0}^\infty$ and fix a basic set Ω so that the generalized finite type condition holds. Let $x \in \mathbb{R}^d$, $0 < b < 1$, and $\mathcal{S} := \{S \in \mathcal{A}_b:\ x \in S(\Omega)\}$. By Lemma 3.1, it suffices to show that there exists $\gamma \in \mathbb{N}$ (independent of x and b) such that

$$\#\mathcal{S} \leqslant \gamma. \tag{3.1}$$

List all elements of $\mathcal{S}$ as $S_{\boldsymbol{i}_1}, \dots, S_{\boldsymbol{i}_m}$. (The choice of the particular $\boldsymbol{i}_j$ does not affect the following proof.) For each $\boldsymbol{i}_j$ there exists a unique $\boldsymbol{i}'_j \in \mathcal{M}_{k_j}$ such that $\boldsymbol{i}'_j \preccurlyeq \boldsymbol{i}_j$. We assume that $\boldsymbol{i}'_j$ is chosen such that k_j is maximum, i.e., if $\boldsymbol{i}^*_j \preccurlyeq \boldsymbol{i}_j$ and $\boldsymbol{i}^*_j \in \mathcal{M}_\ell$ for some ℓ, then $\ell \leqslant k_j$ and $\boldsymbol{i}^*_j \preccurlyeq \boldsymbol{i}'_j$. Assume without loss of generality that

$$k_1 = \min\{k_i:\ 1 \leqslant i \leqslant m\}.$$

Then for each $j \in \{2, \dots, m\}$, there exists $\boldsymbol{i}''_j \in \mathcal{M}_{k_1}$ such that $\boldsymbol{i}''_j \preccurlyeq \boldsymbol{i}'_j \preccurlyeq \boldsymbol{i}_j$. Hence, by letting $\boldsymbol{i}''_1 := \boldsymbol{i}'_1$, we can write

$$\boldsymbol{i}_j = (\boldsymbol{i}'_j, \boldsymbol{l}'_j) = (\boldsymbol{i}''_j, \boldsymbol{l}''_j), \quad j = 1, \dots, m,$$

with $\boldsymbol{i}''_j \in \mathcal{M}_{k_1}$ and $\boldsymbol{i}'_j \in \mathcal{M}_{k_j}$.

Since each $S_{\boldsymbol{i}_j}$ belongs to $\mathcal{A}_b$, we have

$$\rho b|\Omega| < |S_{\boldsymbol{i}_j}(\Omega)| \leqslant b|\Omega|, \quad 1 \leqslant j \leqslant m.$$

Also, by the definition of $\{\mathcal{M}_k\}_{k=0}^\infty$, there exists a constant L, independent of x and b, such that $|\boldsymbol{i}_j| - |\boldsymbol{i}'_j| \leqslant L$ for all $j \in \{1, \dots, m\}$. Hence,

$$\rho b|\Omega| < |S_{\boldsymbol{i}'_j}(\Omega)| \leqslant b\rho^{-L}|\Omega|, \quad 1 \leqslant j \leqslant m.$$

Combining the above estimations yields, in particular,

$$\rho \leqslant \frac{|S_{\boldsymbol{i}'_1}(\Omega)|}{|S_{\boldsymbol{i}_j}(\Omega)|} \leqslant \rho^{-(L+1)}, \quad 1 \leqslant j \leqslant m. \tag{3.2}$$

Since $x \in S(\Omega)$ for all $S \in \mathcal{S}$, it follows that $\boldsymbol{v}_2 = (S_{\boldsymbol{i}''_2}, k_1), \ldots, \boldsymbol{v}_m = (S_{\boldsymbol{i}''_m}, k_1)$ are neighbors of $\boldsymbol{v}_1 = (S_{\boldsymbol{i}'_1}, k_1)$. The generalized finite type condition implies that the number of members in each neighborhood type is bounded by some constant M independent of x and b, and thus $\#\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\} \leqslant M$, i.e.,

$$\#\{S_{\boldsymbol{i}''_j}\colon\ j = 1, \ldots, m\} \leqslant M. \tag{3.3}$$

It also implies that there exists a constant $C_1 > 0$, independent of x and b, such that

$$C_1^{-1} \leqslant \frac{|S_{\boldsymbol{i}''_j}(\Omega)|}{|S_{\boldsymbol{i}'_1}(\Omega)|} \leqslant C_1, \quad 1 \leqslant j \leqslant m. \tag{3.4}$$

Combining (3.2) and (3.4) yields

$$C_2^{-1} \leqslant \frac{|S_{\boldsymbol{i}''_j}(\Omega)|}{|S_{\boldsymbol{i}_j}(\Omega)|} \leqslant C_2, \quad 1 \leqslant j \leqslant m, \tag{3.5}$$

where $C_2 := \rho^{-(L+1)} C_1$.

For each $j \in \{1, \ldots, m\}$, (3.5) implies, in particular, that

$$\rho_{\boldsymbol{i}''_j} \leqslant \rho_{\boldsymbol{i}_j} C_2 = \rho_{\boldsymbol{i}''_j} \rho_{\boldsymbol{l}''_j} C_2, \quad 1 \leqslant j \leqslant m.$$

Hence for $j \in \{1, \ldots, m\}$, $\rho_{\max}^{|\boldsymbol{l}''_j|} \geqslant \rho_{\boldsymbol{l}''_j} \geqslant C_2^{-1}$ and thus $|\boldsymbol{l}''_j| \leqslant -\ln(C_2)/\ln \rho_{\max}$. Now, if we let $\ell = [-\ln(C_2)/\ln \rho_{\max}] + 1$, then for each $\boldsymbol{i}'' \in \{\boldsymbol{i}''_1, \ldots, \boldsymbol{i}''_m\}$,

$$\#\{S_{\boldsymbol{i}_j}\colon\ \boldsymbol{i}'' \preccurlyeq \boldsymbol{i}_j,\ j = 1, \ldots, m\} \leqslant N^{\ell}.$$

But (3.3) says that there are no more than M distinct $S_{\boldsymbol{i}''}$. Thus, (3.1) follows by taking $\gamma = MN^{\ell}$. □

In general, the WSP does not imply the generalized finite type condition. The following example in [23] serves as a counterexample: $f_i(x) = \rho R_i x$, $i = 1, 2$, where $0 < \rho < 1$ and R_1, R_2 are incommensurable rotations. Since the attractor is the point $\{0\}$, by letting $x_0 = 0$ in Definition 3.1, we see that the IFS has the WSP. However, the generalized finite type condition fails. In fact, for any sequence of nested index sets $\{\mathcal{M}_k\}_{k=0}^{\infty}$ and any invariant open set Ω chosen, there is always a sequence of neighborhoods whose number of members tends to infinity. We do not know if the WSP and the generalized finite type condition are equivalent if we assume that the attractor does not lie in a hyperplane.

4. Proof of the dimension formula

In this section we assume that $\{S_i\}_{i=1}^N$ is of generalized finite type and fix a sequence of nested index sets $\{\mathcal{M}_k\}_{k=1}^\infty$ and a basic set Ω. Let $\mathcal{T}_1, \dots, \mathcal{T}_q$ denote all the distinct neighborhood types, with $\mathcal{T}_1 = [\boldsymbol{v}_{\mathrm{root}}]$. For each $\alpha \geqslant 0$ we define a *weighted incidence matrix* $A_\alpha = (A_\alpha(i,j))_{i,j=1}^q$ as follows. Fix i ($1 \leqslant i \leqslant q$) and a vertex $\boldsymbol{v} \in \mathcal{V}_R$ such that $[\boldsymbol{v}] = \mathcal{T}_i$. Let $\boldsymbol{u}_1, \dots, \boldsymbol{u}_m$ be the offspring of $\boldsymbol{v}$ in $\mathcal{G}_R$ and let $\boldsymbol{i}_k$, $1 \leqslant k \leqslant m$, be the unique edge in $\mathcal{G}_R$ connecting $\boldsymbol{v}$ to $\boldsymbol{u}_k$. Then we define

$$A_\alpha(i,j) := \sum \bigl\{ \rho_{\boldsymbol{i}_k}^\alpha : \boldsymbol{v} \xrightarrow{\boldsymbol{i}_k} \boldsymbol{u}_k,\ [\boldsymbol{u}_k] = \mathcal{T}_j \bigr\}. \tag{4.1}$$

According to Proposition 2.4, the definition of A_α is independent of the choice of $\boldsymbol{v}$ above.

The rest of this section is devoted to the proof of Theorem 1.2. We denote by $\boldsymbol{v} \to_R \boldsymbol{u}$ if $\boldsymbol{v}, \boldsymbol{u} \in \mathcal{V}_R$ and $\boldsymbol{u}$ is an offspring of $\boldsymbol{v}$ in the reduced graph $\mathcal{G}_R$. We define a *path* in $\mathcal{G}_R$ to be an infinite sequence $(\boldsymbol{v}_0, \boldsymbol{v}_1, \dots)$ such that $\boldsymbol{v}_k \in \mathcal{V}_k$ and $\boldsymbol{v}_k \to_R \boldsymbol{v}_{k+1}$ for all $k \geqslant 0$, with $\boldsymbol{v}_0 = \boldsymbol{v}_{\mathrm{root}}$.

Let $\mathcal{P}$ be the set of all paths in $\mathcal{G}_R$. If the vertices $\boldsymbol{v}_0 = \boldsymbol{v}_{\mathrm{root}}, \boldsymbol{v}_1, \dots, \boldsymbol{v}_k$ are such that $\boldsymbol{v}_j \to_R \boldsymbol{v}_{j+1}$ for $1 \leqslant j \leqslant k-1$, we call the set

$$I_{\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_k} := \bigl\{ (\boldsymbol{u}_0, \boldsymbol{u}_1, \dots) \in \mathcal{P}\colon\ \boldsymbol{u}_j = \boldsymbol{v}_j \text{ for all } 0 \leqslant j \leqslant k \bigr\}$$

a *cylinder*. Since the path from $\boldsymbol{v}_0$ to $\boldsymbol{v}_k$ in $\mathcal{G}_R$ is unique, we denote

$$I_{\boldsymbol{v}_k} := I_{\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_k}.$$

We define a measure $\hat{\mu}$ on $\mathcal{P}$. For each cylinder $I_{\boldsymbol{v}_k}$, where $\boldsymbol{v}_k \in \mathcal{V}_k$ and $[\boldsymbol{v}_k] = \mathcal{T}_i$, we let

$$\hat{\mu}(\boldsymbol{v}_{\mathrm{root}}) = a_1 = 1 \quad \text{and} \quad \hat{\mu}(I_{\boldsymbol{v}_k}) = \rho_{\boldsymbol{v}_k}^\alpha a_i.$$

where $[a_1, \dots, a_q]^T$ is a 1-eigenvector of A_α, normalized so that $a_1 = 1$ (this is possible because all neighborhood types are descendants of $\mathcal{T}_1$).

To show that $\hat{\mu}$ is indeed a measure on $\mathcal{P}$, we notice that two cylinders $I_{\boldsymbol{v}}$ and $I_{\boldsymbol{v}'}$ with $\boldsymbol{v} \in \mathcal{V}_k$, $\boldsymbol{v}' \in \mathcal{V}_\ell$ and $k \leqslant \ell$, intersect if and only if either $\boldsymbol{v}' = \boldsymbol{v}$ in the case $k = \ell$ or $\boldsymbol{v}'$ is a descendant of $\boldsymbol{v}$ in the case $k < \ell$. In both cases, $I_{\boldsymbol{v}'} \subseteq I_{\boldsymbol{v}}$. Now let $\boldsymbol{v} \in \mathcal{V}_R$ and let $\mathcal{D}$ denote the set of all offspring of $\boldsymbol{v}$ in $\mathcal{G}_R$. Then

$$\begin{aligned}
\sum_{\boldsymbol{u} \in \mathcal{D}} \hat{\mu}(I_{\boldsymbol{u}}) &= \sum_{j=1}^q \sum \bigl\{ \hat{\mu}(I_{\boldsymbol{u}})\colon\ \boldsymbol{u} \in \mathcal{D},\ [\boldsymbol{u}] = \mathcal{T}_j \bigr\} \\
&= \sum_{j=1}^q \sum \bigl\{ \rho_{\boldsymbol{u}}^\alpha a_j\colon\ \boldsymbol{u} \in \mathcal{D},\ [\boldsymbol{u}] = \mathcal{T}_j \bigr\} \\
&= \rho_{\boldsymbol{v}}^\alpha \sum_{j=1}^q \sum \bigl\{ \rho_{\boldsymbol{i}_k}^\alpha a_j : \boldsymbol{v} \xrightarrow{\boldsymbol{i}_k}{}_R \boldsymbol{u},\ \boldsymbol{u} \in \mathcal{D},\ [\boldsymbol{u}] = \mathcal{T}_j \bigr\} \\
&= \rho_{\boldsymbol{v}}^\alpha \sum_{j=1}^q A_\alpha(i,j) a_j \\
&= \rho_{\boldsymbol{v}}^\alpha a_i = \hat{\mu}(I_{\boldsymbol{v}}).
\end{aligned}$$

It follows now from $\hat{\mu}(\mathcal{P}) = \hat{\mu}(\boldsymbol{v}_{\text{root}}) = 1$ that $\hat{\mu}$ is indeed a measure on $\mathcal{P}$.

For any bounded Borel set $E \subseteq \mathbb{R}^d$, let $\mathcal{B} = \mathcal{B}(E)$ be defined as

$$\mathcal{B}(E) := \big\{ I_{\boldsymbol{v}_k} = I_{\boldsymbol{v}_0,\dots,\boldsymbol{v}_k} \colon \big|S_{\boldsymbol{v}_k}(\Omega)\big| \leqslant |E| < \big|S_{\boldsymbol{v}_{k-1}}(\Omega)\big| \text{ and } E \cap S_{\boldsymbol{v}_k}(\Omega) \neq \emptyset \big\}. \tag{4.2}$$

Lemma 4.1. *Let Ω be a basic set as above. Then there exists a constant $C_0 > 0$, independent of k, such that for any bounded Borel set $E \subseteq \mathbb{R}^d$, $\#\mathcal{B}(E) \leqslant C_0$.*

Proof. Since there is a one-to-one correspondence between $I_{\boldsymbol{v}_k}$ and $\boldsymbol{v}_k$, we have $\#\mathcal{B} = \#\widetilde{\mathcal{B}}$, where

$$\begin{aligned} \widetilde{\mathcal{B}} &:= \big\{ \boldsymbol{v}_k \in \mathcal{V}_k \colon \big|S_{\boldsymbol{v}_k}(\Omega)\big| \leqslant |E| < \big|S_{\boldsymbol{v}_{k-1}}(\Omega)\big| \text{ and } E \cap S_{\boldsymbol{v}_k}(\Omega) \neq \emptyset \big\} \\ &= \big\{ \boldsymbol{v}_k \in \mathcal{V}_k \colon \rho_{\boldsymbol{v}_k} \leqslant |E|/|\Omega| < \rho_{\boldsymbol{v}_{k-1}} \text{ and } E \cap S_{\boldsymbol{v}_k}(\Omega) \neq \emptyset \big\}. \end{aligned}$$

Let $b := |E|/|\Omega|$ and let $\boldsymbol{v}_k \in \widetilde{\mathcal{B}}$. Then there exists a unique $\boldsymbol{i} \in \mathcal{M}_k$ such that $\boldsymbol{v}_k = (S_{\boldsymbol{i}}, k)$. The index $\boldsymbol{i}$ is unique because we are considering $\boldsymbol{v}_k$ to be in the reduced graph $\mathcal{G}_R$. Let $\boldsymbol{i}' \preccurlyeq \boldsymbol{i}$ such that $S_{\boldsymbol{i}'} \in \mathcal{A}_b$. Condition (5) of the definition of $\{\mathcal{M}_k\}_{k=0}^{\infty}$ and the inequalities $\rho_{\boldsymbol{i}'} \leqslant b < \rho_{\boldsymbol{v}_{k-1}}$ together imply that $|\boldsymbol{i}| - |\boldsymbol{i}'| \leqslant L$.

Fix any $x_0 \in E$. Then the assumption $E \cap S_{\boldsymbol{v}_k}(\Omega) \neq \emptyset$ implies that $E \cap S_{\boldsymbol{i}'}(\Omega) \neq \emptyset$. Moreover, $|S_{\boldsymbol{i}'}(\Omega)| \leqslant b|\Omega|$ since $S_{\boldsymbol{i}'} \in \mathcal{A}_b$. Thus, $S_{\boldsymbol{i}'}(\Omega) \subseteq B_\delta(x_0)$, where $\delta := 2b|\Omega|$. The generalized finite type condition implies that there exists a constant $\gamma > 0$, independent of b, such that for all $x \in \mathbb{R}^d$,

$$\#\big\{ S \in \mathcal{A}_b \colon x \in S(\Omega) \big\} \leqslant \gamma$$

(see the proof of Theorem 1.1). Hence, as in the proof of Lemma 3.1, we have

$$(b\rho)^d \mathcal{L}(\Omega) \#\big\{ S_{\boldsymbol{i}'} \colon E \cap S_{\boldsymbol{i}'}(\Omega) \neq \emptyset \big\} \leqslant \sum \big\{ \mathcal{L}\big(S_{\boldsymbol{i}'}(\Omega)\big) \colon E \cap S_{\boldsymbol{i}'}(\Omega) \neq \emptyset \big\} \leqslant \gamma \mathcal{L}\big(B_\delta(x_0)\big)$$

and thus

$$\#\big\{ S_{\boldsymbol{i}'} \colon E \cap S_{\boldsymbol{i}'}(\Omega) \neq \emptyset \big\} \leqslant \frac{(2|\Omega|)^d \gamma c_d}{\rho^d \mathcal{L}(\Omega)} := C_1,$$

where c_d is the volume of the unit ball in $\mathbb{R}^d$. Consequently,

$$\#\mathcal{B} = \#\widetilde{\mathcal{B}} \leqslant N^L \#\big\{ S_{\boldsymbol{i}'} \colon E \cap S_{\boldsymbol{i}'}(\Omega) \neq \emptyset \big\} \leqslant C_1 N^L,$$

proving the lemma. □

Proof of Theorem 1.2. Since F is a self-similar set, $\dim_{\mathrm{B}}(F) = \dim_{\mathrm{H}}(F)$ (see [4]). We will prove that $\mathcal{H}^\alpha(F) > 0$ and $\dim_{\mathrm{H}}(F) = \alpha$. They imply that $\mathcal{H}^\alpha(F) < \infty$ since F is self-similar (see [5] or [7]). Note that they are true if F does not lie in a hyperplane (see Remark 4.2 following the proof), but we are not able to use these facts in the general case here. The proof makes use of some ideas in [18,20].

Lower bound. To prove the lower bound, we will transfer the measure $\hat{\mu}$ constructed above to a measure on F. Observe that by assumptions (3) and (4) in the definition of $\{\mathcal{M}_k\}_{k=0}^{\infty}$, for all $k \geqslant 0$,

$$F = \bigcup_{\boldsymbol{v} \in \mathcal{V}_k \cap \mathcal{V}_R} S_{\boldsymbol{v}}(F).$$

Define $f : \mathcal{P} \to \mathbb{R}^d$ by letting $f(\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots)$ be the unique point in $\bigcap_{k=0}^{\infty} S_{\boldsymbol{v}_k}(F)$. It is clear that $f(\mathcal{P}) = F$. Now define $\mu := \hat{\mu} \circ f^{-1}$, where μ is the measure on $\mathcal{P}$ defined at the beginning of this section.

Let E be a bounded Borel subset of $\mathbb{R}^d$ and let $\mathcal{B} = \mathcal{B}(E)$ be defined as in (4.2). Note that

$$\mu(E) \leqslant \sum_{I_{\boldsymbol{v}_k} \in \mathcal{B}} \hat{\mu}(I_{\boldsymbol{v}_k}).$$

Also, if we assume that $[\boldsymbol{v}_k] = \mathcal{T}_{i_k}$. Then

$$\frac{\hat{\mu}(I_{\boldsymbol{v}_k})}{|S_{\boldsymbol{v}_k}(\Omega)|^{\alpha}} = \frac{\rho_{\boldsymbol{v}_k}^{\alpha} a_{i_k}}{\rho_{\boldsymbol{v}_k}^{\alpha} |\Omega|^{\alpha}} = \frac{a_{i_k}}{|\Omega|^{\alpha}}.$$

Hence,

$$\mu(E) \leqslant \sum_{I_{\boldsymbol{v}_k} \in \mathcal{B}} \frac{a_{i_k} |S_{\boldsymbol{v}_k}(\Omega)|^{\alpha}}{|\Omega|^{\alpha}} \leqslant |E|^{\alpha} \sum_{I_{\boldsymbol{v}_k} \in \mathcal{B}} \frac{a_{i_k}}{|\Omega|^{\alpha}} \leqslant |E|^{\alpha} \max_{1 \leqslant i \leqslant q} \left\{ \frac{a_i}{|\Omega|^{\alpha}} \right\} \#\mathcal{B}.$$

By Lemma 4.1, $\#\mathcal{B}$ is bounded by a constant independent of E. Hence $\mu(E) \leqslant C|E|^{\alpha}$ for some constant $C > 0$. Thus, $\mathcal{H}^{\alpha}(F) > 0$ and $\dim_{\mathrm{H}}(F) \geqslant \alpha$ (see [5]), which is the required lower bound.

Upper bound. To obtain the upper bound $\dim_{\mathrm{H}}(F) \leqslant \alpha$, we first assume that A_{α} is irreducible and thus all the a_i's are positive. Since $F \subseteq \overline{\Omega}$, for each $k \geqslant 0$,

$$F \subseteq \bigcup_{\boldsymbol{v}_k \in \mathcal{V}_k \cap \mathcal{V}_R} S_{\boldsymbol{v}_k}(\overline{\Omega}).$$

Moreover,

$$\begin{aligned}
\sum_{\boldsymbol{v}_k \in \mathcal{V}_k \cap \mathcal{V}_R} \left| S_{\boldsymbol{v}_k}(\overline{\Omega}) \right|^{\alpha} &= \sum_{\boldsymbol{v}_k \in \mathcal{V}_k \cap \mathcal{V}_R} \rho_{\boldsymbol{v}_k}^{\alpha} |\overline{\Omega}|^{\alpha} \\
&= \sum_{i=1}^{q} \sum \left\{ \rho_{\boldsymbol{v}_k}^{\alpha} a_i \frac{|\overline{\Omega}|^{\alpha}}{a_i} \colon \boldsymbol{v}_k \in \mathcal{V}_k \cap \mathcal{V}_R,\ [\boldsymbol{v}_k] = \mathcal{T}_i \right\} \\
&\leqslant \left(\max_{1 \leqslant i \leqslant q} \frac{|\overline{\Omega}|^{\alpha}}{a_i} \right) \sum_{\boldsymbol{v}_k \in \mathcal{V}_k \cap \mathcal{V}_R} \hat{\mu}(I_{\boldsymbol{v}_k}) \\
&\leqslant \max_{1 \leqslant i \leqslant q} \frac{|\overline{\Omega}|^{\alpha}}{a_i} < \infty.
\end{aligned}$$

Since $\{S_{\boldsymbol{v}_k}(\overline{\Omega})\colon \boldsymbol{v}_k \in \mathcal{V}_k \cap \mathcal{V}_R\}$ is a cover of F and $\limsup_{k\to\infty}\{|S_{\boldsymbol{v}_k}(\overline{\Omega})|\colon \boldsymbol{v}_k \in \mathcal{V}_k\} = 0$ by contractivity, the definition of Hausdorff measure implies that $\mathcal{H}^\alpha(F) < \infty$, and thus $\dim_{\mathrm{H}}(F) \leqslant \alpha$.

Now assume A_α is not irreducible. After a suitable permutation of the neighborhood types, we can assume that A_α has the form

$$A_\alpha = \begin{bmatrix} A_1 & * & \dots & * \\ 0 & * & \dots & * \\ \vdots & & \ddots & * \\ 0 & \dots & \dots & A_m \end{bmatrix},$$

where each A_i is either an irreducible square matrix or a 1×1 zero matrix (see, e.g., [1]). Let

$$\mathcal{E} := \{A_i\colon 1 \leqslant i \leqslant m\}, \quad \text{counting multiplicity,}$$

$$\mathcal{E}^* := \{A_i \in \mathcal{E}\colon A_i \text{ is nonzero}\}, \quad \text{counting multiplicity.}$$

For $A_i \in \mathcal{E}$, let $\mathcal{T}_{A_i}$ be the collection of neighborhood types corresponding to A_i. Note that for $i \neq j$, $\mathcal{T}_{A_i} \neq \mathcal{T}_{A_j}$ (even though it is possible that $A_i = A_j$ as matrices). This is the reason to count multiplicity when defining $\mathcal{E}$ and $\mathcal{E}^*$.

Each $A_i \in \mathcal{E}^*$ clearly corresponds to a subset $F_{A_i} \subseteq F$ defined as follows

$$F_{A_i} := \bigcap_{k=1}^{\infty} \bigcup \bigl\{S_{\boldsymbol{u}_1} \circ \cdots \circ S_{\boldsymbol{u}_k}(F)\colon [\boldsymbol{u}_1], \dots, [\boldsymbol{u}_k] \in \mathcal{T}_{A_i},\ \boldsymbol{u}_j \to_R \boldsymbol{u}_{j+1},\ j = 1, \dots, k-1\bigr\}.$$

Obviously, the proof of the irreducible case above yields $\mathcal{H}^\alpha(F_{A_i}) < \infty$.

For each $A_i \in \mathcal{E}^*$ and $k \geqslant 0$ define

$$\mathcal{P}_{A_i}(k) := \bigl\{(\boldsymbol{v}_0, \boldsymbol{v}_1, \dots) \in \mathcal{P}\colon [\boldsymbol{v}_{k-1}] \notin \mathcal{T}_{A_i} \text{ and } [\boldsymbol{v}_\ell] \in \mathcal{T}_{A_i}\ \forall \ell \geqslant k\bigr\}.$$

In view of the decomposition of A_α, for each path $(\boldsymbol{v}_0, \boldsymbol{v}_1, \dots) \in \mathcal{P}$, if $k \geqslant 0$ is such that $[\boldsymbol{v}_k] \in \mathcal{T}_{A_i}$, then for $\ell \geqslant k$, either $[\boldsymbol{v}_\ell]$ belongs to $\mathcal{T}_{A_i}$ for all $\ell \geqslant k$, or there exists some $\ell_0 > k$ such that $[\boldsymbol{v}_{\ell_0}] \in \bigcup_{j>i} \mathcal{T}_{A_j}$. Repeating this argument, we see that each path in $\mathcal{P}$ must belong to some $\mathcal{P}_{A_i}(k)$ for some $A_i \in \mathcal{E}^*$ and $k \geqslant 0$. Hence we can partition the set of all paths $\mathcal{P}$ as

$$\mathcal{P} = \bigcup_{A_i \in \mathcal{E}^*} \bigcup_{k=0}^{\infty} \mathcal{P}_{A_i}(k).$$

Consequently,

$$F = f(\mathcal{P}) = \bigcup_{A_i \in \mathcal{E}^*} \bigcup_{k=0}^{\infty} f\bigl(\mathcal{P}_{A_i}(k)\bigr).$$

For each $A_i \in \mathcal{E}^*$ and $k \geqslant 0$, the definitions of f and F_{A_i} imply that

$$f\bigl(\mathcal{P}_{A_i}(k)\bigr) \subseteq \bigcup_{\boldsymbol{v}_0, \dots, \boldsymbol{v}_{k-1}} S_{\boldsymbol{v}_0} \circ \cdots \circ S_{\boldsymbol{v}_{k-1}}(F_{A_i}),$$

where the union is over all vertices $\boldsymbol{v}_0, \dots, \boldsymbol{v}_{k-1} \in \mathcal{V}_R$ such that $\boldsymbol{v}_j \to_R \boldsymbol{v}_{j+1}$ for $j = 0, \dots, k-2$. It follows that

$$\mathcal{H}^\alpha\big(f\big(\mathcal{P}_{A_i}(k)\big)\big) \leqslant \sum_{\boldsymbol{v}_0,\dots,\boldsymbol{v}_{k-1}} \mathcal{H}^\alpha\big(S_{\boldsymbol{v}_0} \circ \cdots \circ S_{\boldsymbol{v}_{k-1}}(F_{A_i})\big)$$
$$\leqslant \sum_{\boldsymbol{v}_0,\dots,\boldsymbol{v}_{k-1}} \rho_{\boldsymbol{v}_0}^\alpha \cdots \rho_{\boldsymbol{v}_{k-1}}^\alpha \mathcal{H}^\alpha(F_{A_i}) < \infty.$$

Hence, $\dim_{\mathrm{H}}(f(\mathcal{P}_{A_i}(k))) \leqslant \alpha$ for all $A \in \mathcal{E}^*$ and $k \geqslant 0$. Finally, it follows from the countable stability of the Hausdorff dimension (see [5]) that $\dim_{\mathrm{H}}(F) \leqslant \alpha$. This completes the proof of the theorem. □

Remark 4.2. Let F be the attractor of an IFS of generalized finite type and α be the Hausdorff dimension of F. Suppose F does not lie in a hyperplane. Then it follows immediately from Theorem 1.1 and [23, Corollary] that $0 < \mathcal{H}^\alpha(F) < \infty$. Theorem 1.2 sharpens this result by removing the assumption that F does not lie in a hyperplane.

5. Examples on computing dimension

In this section we illustrate Theorem 1.2 by some examples. We first introduce a way to denote symbolically how the neighborhood types are generated.

As in the previous section we assume that $\{S_i\}_{i=1}^N$ is of generalized finite type and let $\mathcal{T}_1, \dots, \mathcal{T}_q$ denote all the distinct neighborhood types, with $\mathcal{T}_1 = [\boldsymbol{v}_{\mathrm{root}}]$. Fix i $(1 \leqslant i \leqslant q)$ and a vertex $\boldsymbol{v} \in \mathcal{V}_R$ such that $[\boldsymbol{v}] = \mathcal{T}_i$. Let $\boldsymbol{u}_1, \dots, \boldsymbol{u}_m$ be the offspring of $\boldsymbol{v}$ in $\mathcal{G}_R$, let $\boldsymbol{i}_k$, $1 \leqslant k \leqslant m$, be the unique edge in $\mathcal{G}_R$ connecting $\boldsymbol{v}$ to $\boldsymbol{u}_k$, and let $C_{ij} := \{\boldsymbol{u}_k\colon 1 \leqslant k \leqslant m, [\boldsymbol{u}_k] = \mathcal{T}_j\}$. Note that for two edges $\boldsymbol{i}_k$ and $\boldsymbol{i}_{k'}$ connecting $\boldsymbol{v}$ to two distinct $\boldsymbol{u}_k$ and $\boldsymbol{u}_{k'}$ with $[\boldsymbol{u}_k] = [\boldsymbol{u}_{k'}] = \mathcal{T}_j$, it is possible that the contractions $\rho_{\boldsymbol{i}_k}$ and $\rho_{\boldsymbol{i}_{k'}}$ are different. (We can see such a possibility easily by taking an IFS satisfying the open set condition and with contraction ratios that are not all equal. See the illustration below.) By partitioning C_{ij} according to $\rho_{\boldsymbol{i}_k}$, we can write $C_{ij} := C_{ij}(1) \cup \cdots \cup C_{ij}(n_{ij})$ such that

$$C_{ij}(\ell) := \{\boldsymbol{u}_k \in C_{ij}\colon \rho_{\boldsymbol{i}_k} = \rho_{ij\ell}\},$$

where $\rho_{ij\ell}$, $\ell = 1, \dots, n_{ij}$, are distinct. Thus, we can express the matrix entry $A_\alpha(i, j)$ defined in (4.1) as

$$A_\alpha(i, j) = \sum_{\ell=1}^{n_{ij}} \#C_{ij}(\ell)\rho_{ij\ell}^\alpha.$$

We can also write symbolically

$$\mathcal{T}_i \longrightarrow \sum_{j=1}^{q} \sum_{\ell=1}^{n_{ij}} \#C_{ij}(\ell)\mathcal{T}_j(\rho_{ij\ell}), \tag{5.1}$$

where the $\mathcal{T}_j(\rho_{ij\ell})$ are defined in an obvious way. We say that $\mathcal{T}_i$ *generates* $\#C_{ij}(\ell)$ neighborhoods of type $\mathcal{T}_j$ with contraction $\rho_{ij\ell}$. We will illustrate this in the examples in this section.

We begin with an IFS $\{S_i\}_{i=1}^N$ satisfying the open set condition. Let ρ_i be the contraction ratio of S_i. According to Example 2.5 all vertices are of the same neighborhood type $\mathcal{T}_1 = [\boldsymbol{v}_{\mathrm{root}}]$. Moreover, $\boldsymbol{v}_{\mathrm{root}}$ generates N vertices $\boldsymbol{v}_i$, connected by the edge $i \in \Sigma_1 = \{1, \ldots, N\}$. Each $\boldsymbol{v}_i$ is of neighborhood type $\mathcal{T}_1$. By (5.1)

$$\mathcal{T}_1 \longrightarrow \mathcal{T}_1(\rho_1) + \cdots + \mathcal{T}_1(\rho_N).$$

Hence the matrix defined by (4.1) becomes

$$A_\alpha = \left[\rho_1^\alpha + \cdots + \rho_N^\alpha\right]$$

and thus $\dim_{\mathrm{H}}(F)$ is the solution of the equation $\sum_{i=1}^N \rho_i^\alpha = 1$, agreeing with the classical formula (1.2). Moreover, $0 < \mathcal{H}^\alpha(F) < \infty$.

To illustrate the symbolic notation in (5.1) above, we take any IFS $\{S_i\}_{i=1}^3$ on $\mathbb{R}^d$ satisfying the open set condition and has $\rho_1 = \rho_2 = r$ and $\rho_3 = s$. Then from the above discussions we have

$$\mathcal{T}_1 \longrightarrow 2\mathcal{T}_1(r) + \mathcal{T}_1(s).$$

For an IFS satisfying the finite type condition, by taking $\mathcal{M}_k$ to be the Λ_k defined in Example 2.2, one can see that Theorem 1.2 implies [20, Theorem 1.1].

We now consider the example in [16] again.

Example 5.1. Let F be the attractor of the IFS $\{S_i\}_{i=1}^3$ on $\mathbb{R}$ defined in (1.3). Example 2.8 shows that it is of generalized finite type. Using Theorem 1.2, we claim that $\dim_{\mathrm{B}}(F) = \dim_{\mathrm{H}}(F) = \alpha$, where α is the unique solution of the equation

$$\rho^\alpha + 2r^\alpha - (\rho r)^\alpha = 1.$$

Moreover, $0 < \mathcal{H}^\alpha(F) < \infty$.

We remark that the results of this example have been obtained in [16]. The dimension formula was obtained there by a more complicated argument.

Proof. We adopt the same setup and notation of Example 2.8, with $\mathcal{T}_1 = [\boldsymbol{v}_{\mathrm{root}}]$, $\mathcal{T}_2 = [(S_1, 1)]$ and $\mathcal{T}_3 = [(S_2, 1)]$. By using the proof and the construction of the reduced graph there, we have (see Fig. 1)

$$\begin{aligned}
\mathcal{T}_1 &\longrightarrow \mathcal{T}_1(r) + \mathcal{T}_2(\rho) + \mathcal{T}_3(r), \\
\mathcal{T}_2 &\longrightarrow \mathcal{T}_2(\rho) + \mathcal{T}_3(r), \\
\mathcal{T}_3 &\longrightarrow \mathcal{T}_1(r) + \mathcal{T}_2(\rho) + \mathcal{T}_3(r).
\end{aligned}$$

It follows that

$$A_\alpha = \begin{bmatrix} r^\alpha & \rho^\alpha & r^\alpha \\ 0 & \rho^\alpha & r^\alpha \\ r^\alpha & \rho^\alpha & r^\alpha \end{bmatrix} := \begin{bmatrix} b & a & b \\ 0 & a & b \\ b & a & b \end{bmatrix},$$

where $a := \rho^\alpha$ and $b = r^\alpha$. Setting the spectral radius of A_α equal to 1 yields

$$\frac{1}{2}\left(a+2b+\sqrt{a^2+4b^2}\right)=1 \quad \Leftrightarrow \quad a+2b-ab=1 \quad \Leftrightarrow \quad \rho^\alpha+2r^\alpha-(\rho r)^\alpha=1.$$

The assertions now follow from Theorem 1.2. □

Example 5.1 has various higher-dimensional generalizations. The following example is a simple two-dimensional extension.

Example 5.2. Let $\{S_i\}_{i=1}^4$ be an IFS on $\mathbb{R}^2$ defined as

$$S_1(\boldsymbol{x})=\rho\boldsymbol{x}, \qquad S_2(\boldsymbol{x})=r\boldsymbol{x}+(\rho-\rho r,0),$$
$$S_3(\boldsymbol{x})=r\boldsymbol{x}+(1-r,0), \qquad S_4(\boldsymbol{x})=r\boldsymbol{x}+(0,1-r),$$

where $0<\rho<1$, $0<r<1$, and $\rho+2r-\rho r\leqslant 1$. Then $\{S_i\}_{i=1}^4$ is of generalized finite type and it does not satisfy the open set condition. Let F be the attractor of the IFS. Then $\dim_{\mathrm{B}}(F)=\dim_{\mathrm{H}}(F)=\alpha$, where α is the unique solution of the equation

$$\rho^\alpha+3r^\alpha-(\rho r)^\alpha=1.$$

Moreover, $0<\mathcal{H}^\alpha(F)<\infty$.

Proof. Let $\mathcal{M}_k=\Sigma_k$ for $k\geqslant 0$ and let $\Omega=(0,1)\times(0,1)$. Then Ω is invariant under $\{S_i\}_{i=1}^4$. The following proof will show that Ω is a basic set (see Fig. 2(a), (b)).

Let $\boldsymbol{v}_0 := \boldsymbol{v}_{\mathrm{root}}$ and $\mathcal{T}_1 := [\boldsymbol{v}_{\mathrm{root}}]$. Then $\boldsymbol{v}_0$ has four offspring, namely, $\boldsymbol{v}_i=(S_i,1)$, $1\leqslant i\leqslant 4$, with $[\boldsymbol{v}_3]=[\boldsymbol{v}_4]=\mathcal{T}_1$. Let $\mathcal{T}_2:=[\boldsymbol{v}_1]$ and $\mathcal{T}_3:=[\boldsymbol{v}_2]$. Then we have

$$\mathcal{T}_1 \longrightarrow 2\mathcal{T}_1(r)+\mathcal{T}_2(\rho)+\mathcal{T}_3(r).$$

$\boldsymbol{v}_1$ generates three offspring in the reduced graph $\mathcal{G}_R$, namely, $\boldsymbol{v}_5=(S_1S_1,2)$, $\boldsymbol{v}_6=(S_1S_2,2)$ and $\boldsymbol{v}_7=(S_1S_4,2)$, with $[\boldsymbol{v}_5]=\mathcal{T}_2$, $[\boldsymbol{v}_6]=\mathcal{T}_3$ and $[\boldsymbol{v}_7]=\mathcal{T}_1$. (The offspring $(S_1S_3,2)$ in $\mathcal{G}$ is removed when constructing $\mathcal{G}_R$ since $(S_1S_3,2)=(S_2S_1,2)$.) Hence,

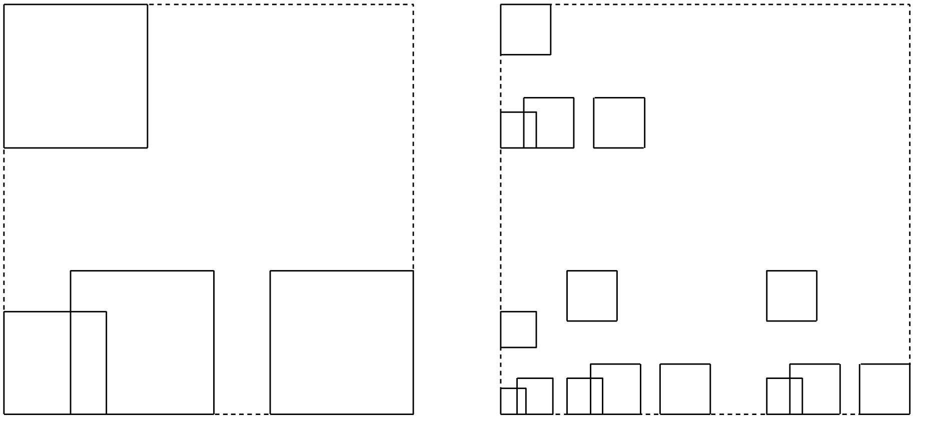
$$\mathcal{T}_2 \longrightarrow \mathcal{T}_1(r)+\mathcal{T}_2(\rho)+\mathcal{T}_3(r).$$

$\boldsymbol{v}_2$ generates four offspring in $\mathcal{G}_R$: $\boldsymbol{v}_8=(S_2S_1,2)$, $\boldsymbol{v}_9=(S_2S_2,2)$, $\boldsymbol{v}_{10}=(S_2S_3,2)$ and $\boldsymbol{v}_{11}=(S_2S_4,2)$, with $[\boldsymbol{v}_8]=\mathcal{T}_2$, $[\boldsymbol{v}_9]=\mathcal{T}_3$, and $[\boldsymbol{v}_{10}]=[\boldsymbol{v}_{11}]=\mathcal{T}_1$. Hence,

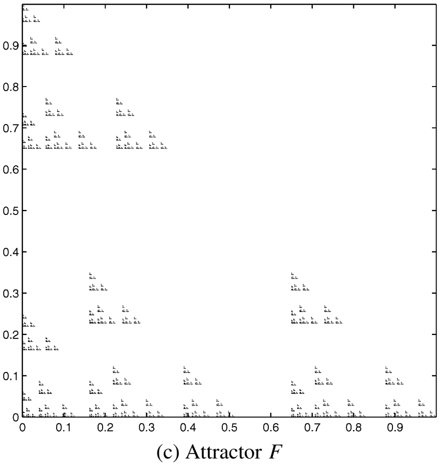
$$\mathcal{T}_3 \longrightarrow 2\mathcal{T}_1(r)+\mathcal{T}_2(\rho)+\mathcal{T}_3(r).$$

Since no new neighborhood types are generated, we conclude that the IFS is of generalized finite type. Moreover,

$$A_\alpha=\begin{bmatrix} 2r^\alpha & \rho^\alpha & r^\alpha \\ r^\alpha & \rho^\alpha & r^\alpha \\ 2r^\alpha & \rho^\alpha & r^\alpha \end{bmatrix} := \begin{bmatrix} 2b & a & b \\ b & a & b \\ 2b & a & b \end{bmatrix},$$



(a) $k = 1$ (b) $k = 2$



(c) Attractor F

Fig. 2. Vertices in $\mathcal{V}_k$ for (a) $k = 1$ and (b) $k = 2$ in Example 5.2, represented by squares. The attractor is shown in (c). The figures are drawn with $\rho = 1/4$ and $r = 7/20$.

with $a := \rho^\alpha$ and $b := r^\alpha$. The spectral radius of A_α is $(a + 3b + \sqrt{a^2 + 2ab + 9b^2})/2$. Setting this equal to 1 yields $a + 3b - ab = 1$. The stated results now follow from Theorem 1.2.

Lastly, $\{S_i\}_{i=1}^4$ does not satisfy the open set condition, since $S_1S_3 = S_2S_1$. □

Figure 2 shows the vertices $\mathcal{V}_k$ for $k = 1, 2$. The box in dotted lines is the set $\Omega = (0, 1) \times (0, 1)$, representing the root vertex $\boldsymbol{v}_{\mathrm{root}}$. In Fig. 2(b), the third square from the left on the bottom row corresponds to two overlapping vertices. Fig. 2(c) shows the attractor F for the case $\rho = 1/4$ and $r = 7/20$ in Example 5.2. In this case, $\dim_{\mathrm{H}}(F)$ is the unique solution of

$$\left(\frac{1}{4}\right)^\alpha + 3\left(\frac{7}{20}\right)^\alpha - \left(\frac{7}{80}\right)^\alpha = 1,$$

which gives $\dim_{\mathrm{H}}(F) = 1.1872563364\ldots.$

Lastly, we remark that in the above examples, we have only used the two different sequences of nested index sets, namely, $\mathcal{M}_k = \Sigma_k$ or $\mathcal{M}_k = \Lambda_k$. We do not have examples that cannot be handled by either of them.

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Appendix A. Example of a vertex that does not have any offspring in the reduced graph $\mathcal{G}_R$

Consider the IFS

$$S_1(x) = \rho x, \qquad S_2(x) = \rho x + \rho^2,$$
$$S_3(x) = \rho x + \rho, \tag{A.1}$$

where $\rho \approx 0.54368899\ldots$, the reciprocal of the Pisot number with minimal polynomial $x^3 - x^2 - x - 1$. Let $\Omega = (0, \rho/(1-\rho)) \approx (0, 1.191487\ldots)$, the interior of the attractor of the IFS. With a sequence of nested index sets $\{\mathcal{M}_k\}$ defined as follows, it can be shown that some vertex in $\mathcal{V}$, denoted by $\boldsymbol{v}_0$ below, does not have any offspring in $\mathcal{G}_R$. The vertex $\boldsymbol{v}_0$ is found with the assistance of a computer search. Let $\boldsymbol{i} = (23322)$, $\boldsymbol{j} = (32111)$, and $\boldsymbol{k} = (31231)$. Define

$$\mathcal{M}_k := \Sigma_k \quad \text{for } k \leqslant 4, \qquad \mathcal{M}_5 := \big(\Sigma_5 \setminus \big\{(23322)\big\}\big) \cup \mathcal{N}_1,$$
$$\mathcal{M}_6 := (\Sigma_6 \setminus \mathcal{N}_1) \cup \mathcal{N}_2, \tag{A.2}$$

where

$$\mathcal{N}_1 := \big\{(\boldsymbol{i}1), (\boldsymbol{i}2), (\boldsymbol{i}3)\big\},$$
$$\mathcal{N}_2 := \big\{(\boldsymbol{i}11), (\boldsymbol{i}12), (\boldsymbol{i}13), (\boldsymbol{i}21), (\boldsymbol{i}22), (\boldsymbol{i}23), (\boldsymbol{i}31), (\boldsymbol{i}32), (\boldsymbol{i}33),$$
$$(\boldsymbol{j}11), (\boldsymbol{j}12), (\boldsymbol{j}13), (\boldsymbol{j}21), (\boldsymbol{j}22), (\boldsymbol{j}23), (\boldsymbol{k}11), (\boldsymbol{k}12), (\boldsymbol{k}13)\big\}.$$

$\mathcal{M}_k$, $k \geqslant 7$, can be easily defined so that $\mathcal{M}_k$, $k \geqslant 0$, is a sequence of nested index sets.

The vertex $\boldsymbol{v}_0 := (\boldsymbol{j}, 5) \in \mathcal{V}_5$ has 13 neighbors (see Fig. 3):

$$\begin{aligned}
&\boldsymbol{v}_1 = ((31212), 5), && \boldsymbol{v}_2 = ((22321), 5), && \boldsymbol{v}_3 = ((23211), 5), && \boldsymbol{v}_4 = ((23123), 5),\\
&\boldsymbol{v}_5 = ((\boldsymbol{i}1), 5), && \boldsymbol{v}_6 = ((\boldsymbol{i}2), 5), && \boldsymbol{v}_7 = ((\boldsymbol{i}3), 5), && \boldsymbol{v}_8 = ((22323), 5),\\
&\boldsymbol{v}_9 = (\boldsymbol{k}, 5), && \boldsymbol{v}_{10} = ((23221), 5), && \boldsymbol{v}_{11} = ((32112), 5), && \boldsymbol{v}_{12} = ((31232), 5),\\
&\boldsymbol{v}_{13} = ((23311), 5).
\end{aligned}$$

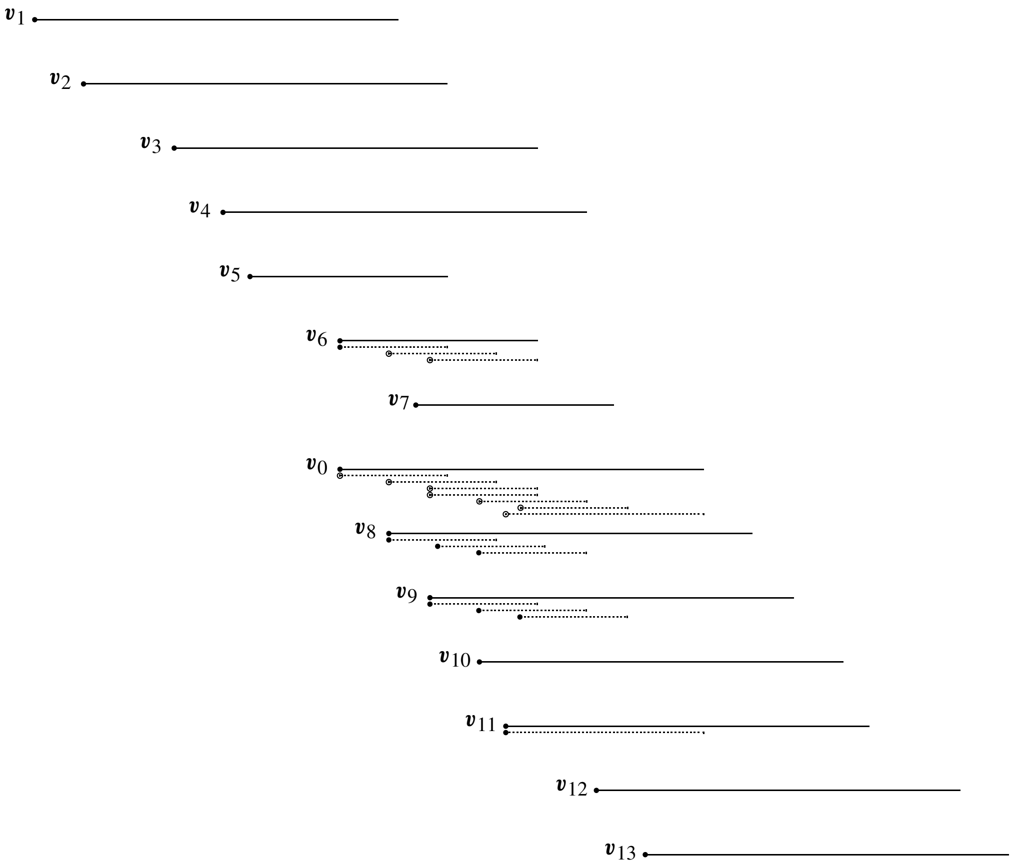

Fig. 3. The vertex $\boldsymbol{v}_0$, all its neighbors, and all the offspring of the neighbors of $\boldsymbol{v}_0$ that overlap with those of $\boldsymbol{v}_0$.

$\boldsymbol{v}_0$ generates the following six offspring in $\mathcal{G}$ that belong to $\mathcal{V}_6$:

$$\boldsymbol{v}_0 \begin{cases} \xrightarrow{(11)} \boldsymbol{u}_{01} = ((\boldsymbol{j}11), 6), \\ \xrightarrow{(12)} \boldsymbol{u}_{02} = ((\boldsymbol{j}12), 6), \\ \xrightarrow{(13),\,(21)} \boldsymbol{u}_{03} = ((\boldsymbol{j}13), 6) = ((\boldsymbol{j}21), 6), \\ \xrightarrow{(21)} \boldsymbol{u}_{04} = ((\boldsymbol{j}22), 6), \\ \xrightarrow{(23)} \boldsymbol{u}_{05} = ((\boldsymbol{j}23), 6), \\ \xrightarrow{(3)} \boldsymbol{u}_{06} = ((\boldsymbol{j}3), 6). \end{cases}$$

Some of the neighbors of $\boldsymbol{v}_0$ generate offspring that coincide with each of the offspring $\boldsymbol{u}_{01}, \dots, \boldsymbol{u}_{06}$ of $\boldsymbol{v}_0$, and have edges that are smaller in the lexicographical order. In fact,

$$\boldsymbol{v}_5 \xrightarrow{(1)} \boldsymbol{u}_{51} = \big((\boldsymbol{i}11), 6\big) = \boldsymbol{u}_{01},$$

$$\boldsymbol{v}_8 \xrightarrow{(11)} \boldsymbol{u}_{81} = \big((2232311), 6\big) = \boldsymbol{u}_{02},$$

$$\boldsymbol{v}_9 \begin{cases} \xrightarrow{(11)} \boldsymbol{u}_{91} = ((\boldsymbol{k}11), 6) = \boldsymbol{u}_{03}, \\ \xrightarrow{(12)} \boldsymbol{u}_{92} = ((\boldsymbol{k}12), 6) = \boldsymbol{u}_{04}, \\ \xrightarrow{(13)} \boldsymbol{u}_{93} = ((\boldsymbol{k}13), 6) = \boldsymbol{u}_{05}, \end{cases}$$

$$\boldsymbol{v}_{11} \xrightarrow{(1)} \boldsymbol{u}_{11,1} = \big((321121), 6\big) = \boldsymbol{u}_{06}.$$

Hence, all offspring of $\boldsymbol{v}_0$ are removed when constructing $\mathcal{G}_R$. Consequently, $\boldsymbol{v}_0$ does not have any offspring in $\mathcal{G}_R$.

The line segments in Fig. 3 are the intervals obtained by the actual iteration of the interval Ω under the IFS in (A.1), with each interval representing a vertex. (For clarity, the intervals are separated vertically.) $\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_{13}$ are the neighbors of $\boldsymbol{v}_0 = (\boldsymbol{j}, 5)$ in $\mathcal{V}_5$; they are drawn using solid line segments. Offspring of these vertices are represented by the dashed line segments. All the offspring of $\boldsymbol{v}_0$ in $\mathcal{V}_6$ are shown. Offspring of the other neighbors of $\boldsymbol{v}_0$ that overlap with the those of $\boldsymbol{v}_0$ are also shown. If two or more offspring are identical, the one connected by an edge which is smallest in the lexicographical order is indicated by a solid dot (at the left end-point) and the other(s) are indicated by a circle (also at the left end-point). Those offspring indicated by a circle are to be removed when constructing the reduced graph. Note that all the offspring of $\boldsymbol{v}_0$ are to be removed. Thus $\boldsymbol{v}_0$ does not generate any offspring in the reduced graph $\mathcal{G}_R$.

Finally, we remark that this cubic Pisot number is used because similar constructions cannot be obtained by using the more familiar golden ratio.

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