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SOME EXCEPTIONAL PHENOMENA IN MULTIFRACTAL FORMALISM: PART I*

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Abstract. Recently it was discovered that the 3-fold convolution of the Cantor measure μ has intricate fractal structure [HL]: the set of local dimensions μ has an isolated point and therefore the standard multifractal formalism does not hold. Our purpose here is to give a detail study of such class of examples and to understand the dilemma. This will shed some light on the multifractal structure of measures arising from iterated functions systems with overlaps, which to a large extend, is still unknown. In this Part I, we concentrate on the L^q-spectrum $\tau(q)$; we give a formula for $\tau(q)$ and show that it is real analytic on $\mathbb{R}$ except for one *non-differentiable* point in $\mathbb{R}^-$. The basic techniques are the product of matrices and the renewal theorem. In Part II, we will prove that such μ satisfies a *modified* multifractal formalism.

Key words. convolution, Cantor measure, dimension, L^q-spectrum, multifractal

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1. Introduction. There are two basic parameters in the study of the multifractal structure of a probability measure μ on $\mathbb{R}^d$: the *dimension spectrum* and the *L^q-spectrum*. The dimension spectrum (also called *singular spectrum*) $f(\alpha)$ of μ is defined as

$$f(\alpha) = \dim_H\{x \in \mathbb{R}^d : \ \alpha(x) = \alpha\},$$

where

$$\alpha(x) = \lim_{\delta\to 0+} \frac{\log \mu\big(B(x,\delta)\big)}{\log \delta}, \qquad x \in \text{supp}\mu,$$

and is called the *local dimension* of μ at x, where $B(x,\delta)$ is the closed ball centered at x with radius δ. The *L^q-spectrum* (also called *moment scaling exponent*) is defined by

$$\tau(q) = \liminf_{\delta\to 0+} \frac{\log S_\delta(q)}{\log \delta}, \tag{1.1}$$

with

$$S_\delta(q) = \sup \sum_i \mu\big(B(x_i,\delta)\big)^q,$$

where $\{B(x_i,\delta)\}_i$ is a countable family of disjoint closed balls centered at $x_i \in \text{supp}\mu$ and the supermum is taken over all such families.

The study of the dimension spectrum was first proposed by physicists to investigate various chaotic models arising from natural phenomena ([FP], [H], [M]). Analogous to the *thermodynamic formalism* in statistical mechanics, they formulated a

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relation for $f(\alpha)$ and $\tau(q)$ by the heuristic principle of *multifractal formalism* through the *Legendre transform*: if the measure μ is constructed from a cascade algorithm and if τ and f are smooth and concave, then

$$\tau^*(\alpha) = f(\alpha) \qquad \text{and} \qquad f^*(q) = \tau(q). \tag{1.2}$$

(Recall that for a concave function g on $\mathbb{R}$, the *Legendre transform* g^* is defined as $g^*(\alpha) = \inf\{q\alpha - g(q) :\ q \in \mathbb{R}\} = \tilde{q}\alpha - g(\tilde{q})$ where $\alpha = g'(\tilde{q})$).

For an *iterated function system* (IFS) $\{S_j\}_{j=1}^N$ of contractive similitudes on $\mathbb{R}^d$, if we associate with a set of probability weights $\{p_j\}_{j=1}^N$, then there exists a unique probability measure μ such that

$$\mu = \sum_{j=1}^N p_j \mu \circ S_j^{-1}, \tag{1.3}$$

[Hut]. We call μ a *self-similar measure*. We say the IFS satisfies the *open set condition* (OSC) if there exists a bounded non-empty open set O such that $S_i(O) \bigcap S_j(O) = \emptyset$ for all $i \neq j$, and $\bigcup_{j=1}^N S_j(O) \subset O$. It is well known that if the IFS satisfies the OSC, then the L^q-spectrum $\tau(q)$ is given explicitly by

$$\sum_{j=1}^N p_j^q r_j^{-\tau(q)} = 1,$$

where r_j is the contraction ratio of S_j ([CM], [LN2]). Furthermore the multifractal formalism (1.2) holds.

If the IFS does not satisfy the OSC (loosely we say the IFS has overlaps), the problem becomes very complicated, both the calculation of $\tau(q)$ and the proof of the multifractal formalism are formidable. To deal with this case, Lau and Ngai [LN2] introduced the notion of *weak separation condition* (WSC) (for simplicity we assume here the similitudes $\{S_j\}_{j=1}^N$ have the same contraction ratio ρ):

There exists $c > 0$ such that for any $n \in \mathbb{N}$ and $J,\ J' \in \{1, 2, \cdots, N\}^n$, then either $S_J(0) = S_{J'}(0)$ or $|S_J(0) - S_{J'}(0)| \geq c\rho^n$, where $S_J = S_{j_1} \circ \cdots S_{j_n}$, $J = j_1 \cdots j_n$.

The reader can refer to Zerner [Z] for various equivalent statements. The condition allows the IFS to have overlaps, but on the other hand it maintains certain separation in the iteration. This class includes many important examples, some typical cases are the $S_j(x) = \beta^{-1}(x + \epsilon_j), \epsilon_j \in \mathbb{Z}$ where β is an integer or a Pisot number (e.g., golden number); more generally in $\mathbb{R}^d$, $S_j(x) = A^{-1}(x + \epsilon_j)$ where A is a $d \times d$ expanding orthogonal matrix with integer entries and $\epsilon_j \in \mathbb{Z}^d$. The singularity and the absolute continuity of the self-similar measures arisen from the IFS with the WSC has been characterized in [LNR] and [LW]. For the multifractal structure, a local version is proved in [LN2]:

THEOREM 1.1. *Suppose $\{S_j\}_{j=1}^N$ satisfies the WSC and let μ be a self-similar measure as in (1.3). If $\tau(q)$ is differentiable at $q_0 > 0$, then μ satisfies the multifractal formalism (1.2) at $\alpha = \tau'(q_0)$.*

The theorem does not guarantee the differentiability of the $\tau(q)$ (note that $\tau(q)$ is concave and hence differentiable with possibly countably many exceptional points), and it offers no information for $q < 0$. In an attempt to understand the situation, Lau and Ngai [LN1] gave a detailed analysis of the $\tau(q),\ q > 0$ for the Bernoulli convolution with a golden ratio contraction (call it *Erdös measure*). It was shown that $\tau(q)$ is analytic for $q > 0$. Feng [F] extended this to $q < 0$ and found a *non-differentiable* point, which is surprising. Nevertheless the multifractal formalism $\tau^*(\alpha) = f(\alpha)$ still holds.

Another surprising result is obtained by Hu and Lau [HL] by inspecting the convolution of the standard Cantor measure ν. The m-fold convolution $\mu = \nu * \cdots * \nu$ satisfies the self-similar identity (1.3) with

$$S_j(x) = \frac{1}{3}(x + 2j) \quad \text{and} \quad p_j = 2^{-m}\binom{m}{j}, \quad j = 0, 1, \cdots, m. \tag{1.4}$$

It is clear that $\{S_j\}_{j=0}^m$ satisfies the OSC when $m = 1, 2$, but instead the WSC when $m \geq 3$. By using a rather complicated combinatoric argument on the multiple representation of the series $\sum_{n=1}^{\infty} \epsilon_n 3^{-n},\ \epsilon_n \in \{1, \cdots, m\}$, it was proved that [HL] for $m = 3$ the local dimension α has range $E = [\underline{\alpha}, \tilde{\alpha}] \cup \{\overline{\alpha}\}$ with

$$\underline{\alpha} = \log_3(8/3) \approx 0.89278, \quad \overline{\alpha} = \log_3 8 \approx 1.89278, \quad \tilde{\alpha} = \log_3(8/\sqrt{b}) \approx 1.1335. \tag{1.5}$$

where $b = \frac{7+\sqrt{13}}{2}$. The result is in contrary to the expectation that the set of local dimensions α (as the domain of the dimension spectrum $f(\alpha)$) is an interval. In this paper we will make a detailed study of this case and clarify the unusual behavior. This together with the Erdös measure will serve as two illuminating examples to study the more general IFS with the WSC.

In this Part I, we will consider the explicit expression of the L^q-spectrum of the 3-fold convolution of the Cantor measure (actually a more general case with the IFS $\{S_j\}_{j=0}^3$ in (1.4) and $\{p_j\}_{j=0}^3$ a set of probability weights). Our technique is to express μ in (1.3) as a vector measure

$$\boldsymbol{\mu}(\cdot) = \sum_{j=0}^{2} T_j \boldsymbol{\mu}(3 \cdot -j) \tag{1.6}$$

on $[0, 1]$ with matrix coefficients T_j defined in (2.2). Note that the new IFS is $\tilde{S}_j(x) = \frac{1}{3}(x + j),\ j = 0, 1, 2$ on $[0, 1]$ which clearly satisfies the OSC. By using the iteration algorithm, we can represent $\boldsymbol{\mu}$ into a product of matrices (Proposition 2.2). This technique has been used extensively in the study of scaling functions in wavelet theory (see [DL1,2], [LWa]). By some simple manipulations of the matrices, we can reduce the above T_0 and T_2 as

$$M_0 = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}.$$

These two matrices actually determine the L^q-spectrum: define

$$\tilde{s}_n(q) = \sum_{J \in \{0,2\}^{n-1}} \|M_J\|^q, \quad n \geq 1$$

where $\|M_J\| = \mathbf{1}^t M_J \mathbf{1}$ with $\mathbf{1}^t = [1, 1]$, $J = j_1, \cdots, j_n$ and $M_J = M_{j_1} \cdots M_{j_n}$. By using the renewal theorem, we prove (Theorem 2.5, 3.6)

THEOREM 1.2. *Let μ be the 3-fold convolution of the Cantor measure. Then the L^q-spectrum is given by*

$$\tau(q) = \begin{cases} q\log_3 8 + \log_3 \tilde{r}(q) & \text{if} \quad q \ge q_0, \\ q \log_3 8 & \text{if} \quad q < q_0, \end{cases}$$

where $\tilde{r}(q)$ satisfies $3^q \sum_{k=0}^{\infty} \tilde{s}_k(q)\tilde{r}(q)^{k+1} = 1$, q_0 satisfies $\tilde{r}(q_0) = 1$ $\quad (q_0 \approx -1.149)$. Furthermore $\tau(q)$ is differentiable except at q_0.

Actually $\tau(q)$ is real analytic for $q > q_0$, following from that of $\tilde{r}(q)$; $\tau(q)$ is linear for $q < q_0$; Also the definition of Legendre transform implies

$$\overline{\alpha} := \sup\{\alpha(x) : x \in \text{supp}\mu\} = \log_3 8 \quad \text{and} \quad \tau^*(\overline{\alpha}) = 0.$$

This is consistent with the conclusion in (1.5) for $\overline{\alpha}$, which is the local dimension of the two end points of supp$\mu = [0, 3]$ [HL]. In fact the conclusion for the set E in (1.5) can be obtained by the matrix product developed here. We will deal with this in Part II [FLW]. For $q > 0$, the expression of $\tau(q)$ has been obtained in [LN3] by a different method; when q is a positive integer, the above series formula for $\tilde{r}(q)$ in Theorem 1.2 can be reduced to finding the root of a polynomial.

It follows from Theorem 1.1 that the multifractal formalism holds for $q > 0$. In Part II, we will consider $q < 0$ and prove a modified multifractal formalism, by taking into consideration of the non-differentiable point q_0 .

The above technique can also be applied to other cases with the WSC. For example, Keane *et al* [KSS] and Pollicot and Simon [PoS] considered the λ-expansion of *deleted* integers; for $\lambda = \frac{1}{3}$, it corresponds to $S_j(x) = \frac{1}{3}(x + j), j = 0, 1, 3$ with weights $p_0 = p_1 = p_3 = \frac{1}{3}$. It satisfies the WSC. By using the above approach it is not hard to find the L^q-spectrum for the corresponding measure. In this case, it is real analytic and the multifractal structure is simple.

The Erdös measure and the 3-fold convolution of the Cantor measure have also been considered by Olivier *et al* [OST] from the point of view of Gibbs property. They showed that the first one is a *weak Gibbs measure* (which implies the validity of the multifractal formalism for $q \in \mathbb{R}$) but the latter is not. For the general case, Feng and Lau [FL] proved that

THEOREM 1.3. *Let $\{M_1, \cdots, M_m\}$ be non-negative $d \times d$ non-negative matrices and let*

$$P(q) = \lim_{n\to\infty} \frac{1}{n} \sum_{|J|=n} \|M_J\|^q, \qquad q > 0.$$

Suppose $\sum_{i=1}^{m} M_i$ is irreducible, then $P(q)$ is differentiable for $q > 0$. Moreover for $\alpha = P'(q)$, $q > 0$,

$$\dim_H \left\{J : \lim_{n\to\infty} \frac{1}{n} \log \|M_J\| = \alpha\right\} = \frac{-P^*(\alpha)}{\log m}.$$

By using Theorems 1.1 and 1.3, we see that if a measure can be put into a self-similar vector-measure (as in (1.6)), then the multifractal formalism holds for

$q > 0$. There is restriction to extend this to $q < 0$ in view of the 3-fold convolution of the Cantor measure. Nevertheless if the matrices have positive entries, then the theorem can be extended to $q < 0$ [FL].

We remark that Shmerkin [Sh] has independently considered the multifractal structure of the 3-fold convolution of the Cantor measure and the extension. His approach is different from ours.

The paper is organized as the follows. In Section 1, we introduce the associated matrices for the measures slightly more general than the 3-fold convolution of the Cantor measure μ. We use the renewal theorem to relate the product of matrices with the $\tau(q)$ and obtain the basic results. In Section 3, we evaluate the special matrices for μ, the 3-fold convolution of the Cantor measure, and prove Theorem 1.2. A similar approach is used to consider the other cases in Section 4 where we make a brief study about the $\frac{1}{3}$-expansion of deleted integers and the other case with contraction ratio N^{-1} for $N \geq 3$.

2. The L^q-spectrum. In this section we will consider the L^q-spectrum of measures slightly more general than the 3-fold convolution of the Cantor measure. Let $S_j(x) = \frac{1}{3}(x+2j),\ j = 0,1,2,3$. It is clear that $K = [0,3]$ is the invariant set of the IFS. We associate the system with probability weights $\{p_j\}_{j=0}^3$, then there exists a unique probability measure μ satisfying

$$\mu(\cdot) = \sum_{j=0}^{3} p_j \mu(3\cdot - 2j). \tag{2.1}$$

By using a technique of Daubechies and Lagarias [DL1,2], we can split the measure μ into a vector-valued measure $\boldsymbol{\mu}$ defined on $\mathbb{R}$:

$$\boldsymbol{\mu}(A) = \begin{bmatrix} \mu\big(A \cap [0,1]\big) \\ \mu\big((A \cap [0,1]) + 1\big) \\ \mu\big((A \cap [0,1]) + 2\big) \end{bmatrix}$$

for any Borel subset $A \subset \mathbb{R}$. It is clear supp $\boldsymbol{\mu} \subset [0,1]$.

Lemma 2.1. *For the measure $\boldsymbol{\mu}$ as above, we have*

$$\boldsymbol{\mu}(A) = \sum_{j=0}^{2} T_j \boldsymbol{\mu}(3A - j), \tag{2.2}$$

where

$$T_0 = \begin{bmatrix} p_0 & 0 & 0 \\ 0 & p_1 & 0 \\ p_3 & 0 & p_2 \end{bmatrix},\ T_1 = \begin{bmatrix} 0 & p_0 & 0 \\ p_2 & 0 & p_1 \\ 0 & p_3 & 0 \end{bmatrix},\ T_2 = \begin{bmatrix} p_1 & 0 & p_0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix}.$$

Proof. Denote by $I = [0,1]$ for short. For the measure μ in (2.1), note that supp$\mu = [0,3]$, hence for any Borel set $A \subset \mathbb{R}$, we have

$$\mu(A \cap I) = \sum_{j=0}^{3} p_j \mu(3A \cap [0,3] - 2j) = p_0\mu(3A \cap [0,3]) + p_1\mu(3A \cap [0,3] - 2).$$

Observe that $\mu(3A \cap [0,3]) = \sum_{i=0}^{2} \mu\big((3A-i) \cap I + i\big)$, and $\mu(3A \cap [0,3] - 2) = \mu\big((3A-2) \cap I\big)$, it follows that

$$\mu(A \cap I) = p_0\mu(3A \cap I) + p_0\mu\big((3A-1) \cap I+1\big) + p_1\mu\big((3A-2) \cap I\big) + p_0\mu\big((3A-2) \cap I+2\big).$$

Similarly

$$\begin{aligned}\mu(A \cap I+1) = p_1\mu(3A \cap I+1) + p_2\mu\big((3A-1) \cap I\big) \\ +p_1\mu\big((3A-1) \cap I+2\big) + p_2\mu\big((3A-2) \cap I+1\big);\end{aligned}$$

$$\mu(A \cap I+2) = p_3\mu(3A \cap I) + p_2\mu\big(3A \cap I+2\big) + p_3\mu\big((3A-1) \cap I+1\big) + p_3\mu\big((3A-2) \cap I+2\big).$$

Writing these to matrix form, we obtain (2.2). ∎

We use $J = j_1 \cdots j_n \in \{0,1,2\}^n$ to denote the multi-index and $|J| = n$ to denote the length of J; write $T_J = T_{j_1} \cdots T_{j_n}$. Define map $\phi_J : \mathbb{R} \to \mathbb{R}$ as following:

$$\phi_J(x) = 3^{-n}x + \sum_{k=1}^{n} 3^{-k} j_k.$$

Let $[\![J]\!]$ be the interval $[\phi_J(0),\ \phi_J(1)] = [\sum_{k=1}^{n} 3^{-k} j_k, 3^{-n} + \sum_{k=1}^{n} 3^{-k} j_k]$, then $[\![J]\!] \subset [0,1]$. With the notation defined above, we can write down the μ-measure of a neighborhood of a point in $[0,3]$.

Proposition 2.2. *For $J \in \{0,1,2\}^n$ and $A \subset [\![J]\!]$, we have $\boldsymbol{\mu}(A) = T_J\boldsymbol{\mu}\big(\phi_J^{-1}(A)\big)$. In particular*

$$\boldsymbol{\mu}([\![J]\!]) = T_J\mathbf{a} \quad \text{and} \quad \mu([\![J]\!]+i) = \mathbf{e}_i^t T_J \mathbf{a}, \quad i = 0,1,2 \tag{2.3}$$

where $\mathbf{a} = \boldsymbol{\mu}([0,1])$, and $\mathbf{e}_i$ is the unit vector in $\mathbb{R}^3$ whose $(i+1)$-st coordinate is 1.

Proof. Note that supp$\boldsymbol{\mu} \subset [0,1]$, hence $\boldsymbol{\mu}(3A - j) = 0$ if $j \neq j_1$. It follows from (2.2) that

$$\boldsymbol{\mu}(A) = T_{j_1}\boldsymbol{\mu}(3A - j_1).$$

Iterating the above identity yields that

$$\boldsymbol{\mu}(A) = T_J\boldsymbol{\mu}(3^n A - 3^{n-1}j_1 - \cdots - j_n) = T_J\boldsymbol{\mu}\big(\phi_J^{-1}(A)\big),$$

This is the first part of the proposition, and (2.3) follows by letting $A = [\![J]\!]$. ∎

In order to calculate $\tau(q)$ in (1.1), we use the formula

$$\tau(q) = -\limsup_{n\to\infty} \frac{1}{n} \log_3 \Big(\sum_{i=0}^{2} \sum_{|J|=n} (\mathbf{e}_i^t T_J \mathbf{1})^q \Big). \tag{2.4}$$

To see this, we use, for positive sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, $x_n \approx y_n$ to denote the existence of $c > 0$ such that $x_n/c \le y_n \le cx_n$ for all $n \in \mathbb{N}$. It follows from Proposition 2.2 that

$$\mu([\![J]\!]+i) \approx \mathbf{e}_i^t T_J \mathbf{1} \quad \text{where} \quad \mathbf{1}^t = [1,1,1]. \tag{2.5}$$

By the result in [LN2], (2.4) is true for $q \geq 0$.

For $q < 0$, let $n \in \mathbb{N}$ be such that $3^{-n} \leq \delta < 3^{-n+1}$ and $\{B(x_i, \delta)\}_i$ be a family of disjoint δ-ball centered at $x_i \in \mathrm{supp}\mu$. For each x_i, there exists $|J_i| = n$ and $j_i^* \in \{0, 1, 2\}$ such that $x_i \in [\![J_i]\!] + j_i^* \subset B(x_i, \delta)$. Hence

$$\sum_i \mu\big(B(x_i, \delta)\big)^q \leq \sum_i \mu([\![J_i]\!] + j_i^*)^q \leq \sum_{i=0}^{2} \sum_{|J|=n} \mu([\![J]\!] + i)^q.$$

This and (2.5) imply that

$$\tau(q) = \liminf_{\delta \to 0+} \frac{\log S_\delta(q)}{\log \delta} \geq -\limsup_{n \to \infty} \frac{1}{n} \log_3 \Big(\sum_{i=0}^{2} \sum_{|J|=n} (\mathbf{e}_i^t T_J \mathbf{1})^q \Big),$$

where $S_\delta(q)$ is as (1.1). For the reverse inequality, we observe that for each $|J| = n-1,\ i = 0, 1, 2$, there exists δ-ball $B_{J,i} \subset [\![J]\!] + i$, and the assertion follows from the same argument as above.

Peres and Solomyak [PS] proved that if μ is a self-similar measure, then the limit in the definition of $\tau(q)$ in (1.1) exists for all $q > 0$. Our proof in the sequel shows that for the present case the limit for $q < 0$ exists also.

Let

$$\bar{T}_0 = \begin{bmatrix} p_0 & 0 \\ p_3 & p_2 \end{bmatrix}, \quad \bar{T}_2 = \begin{bmatrix} p_1 & p_0 \\ 0 & p_3 \end{bmatrix}.$$

In the following, we will reduce the limit (2.4) into a more tractable form.

Lemma 2.3. *For any $J \in \{0, 2\}^n$, $u, v \in \mathbb{R}$, we have*

$$[u, 0, v] T_J T_1 = \Big([u, v] \bar{T}_J \begin{bmatrix} p_0 \\ p_3 \end{bmatrix} \Big) \mathbf{e}_1^t.$$

Proof. We observe that for $J \in \{0, 2\}^n$, T_J is of the form $\left[\begin{smallmatrix} \alpha_1 & 0 & \alpha_2 \\ 0 & \alpha_3 & 0 \\ \alpha_4 & 0 & \alpha_5 \end{smallmatrix}\right]$ and $\bar{T}_J = \left[\begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_4 & \alpha_5 \end{smallmatrix}\right]$. A direct calculation shows the assertion. ∎

In order to calculate $\tau(q)$ in (2.4), we define $u_0 = 1$ and

$$u_n := u_n(q) = \sum_{|J|=n} (\mathbf{e}_1^t T_J \mathbf{1})^q, \quad n \in \mathbb{N}. \tag{2.6}$$

Also we let $s_0 := s_0(q) = p_1^q + p_2^q$ and for $n \geq 1$, we define

$$s_n := s_n(q) = \sum_{J \in \{0,2\}^{n-1}} \Big([p_2, p_1] \bar{T}_J \begin{bmatrix} p_0 \\ p_3 \end{bmatrix} \Big)^q;$$

$$b_n := b_n(q) = \sum_{J \in \{0,2\}^{n-1}} \big([p_2, p_1] \bar{T}_J \mathbf{1} \big)^q.$$

The following lemma gives the relation between u_n, s_n and b_n through the Renewal Theorem [Fe, p.330, Theorem 1].

LEMMA 2.4. *With the notation as above, we have*
(i) $u_n = \sum_{k=0}^{n-1} s_k u_{(n-1)-k} + b_n, \quad n \geq 1.$
(ii) *If for any $q \in \mathbb{R}$, there exists a unique positive number $r := r(q)$ such that*

$$\sum_{k=0}^{\infty} s_k r^{k+1} = 1 \quad \textit{with} \quad \sum_{k=1}^{\infty} k s_k r^{k+1} < \infty \quad \textit{and} \quad \sum_{k=1}^{\infty} r^k b_k < \infty. \tag{2.7}$$

Then $\lim_{n\to\infty} (u_n)^{1/n} = r^{-1}$.

Proof. Write u_n as

$$u_n := I_n^{(0)} + I_n^{(1)} + I_n^{(2)} \quad \text{with} \quad I_n^{(j)} = \sum_{|J|=n-1} (\mathbf{e}_1^t T_j T_J \mathbf{1})^q, \quad j = 0, 1, 2.$$

Note that $\mathbf{e}_1^t T_0 = p_1 \mathbf{e}_1^t$ and $\mathbf{e}_1^t T_2 = p_2 \mathbf{e}_1^t$, we have $I_n^{(0)} = p_1^q u_{n-1}$ and $I_n^{(2)} = p_2^q u_{n-1}$. By Lemma 2.3,

$$\begin{aligned}
I_n^{(1)} &= \sum_{|J|=n-1} ([p_2, 0, p_1] T_J \mathbf{1})^q \\
&= \sum_{|J|=n-2} ([p_2, 0, p_1] T_1 T_J \mathbf{1})^q + \sum_{|J|=n-2} \sum_{i=0,2} ([p_2, 0, p_1] T_i T_J \mathbf{1})^q \\
&= \sum_{|J|=n-2} ([p_2, p_1] \begin{bmatrix} p_0 \\ p_3 \end{bmatrix} \mathbf{e}_1^t T_J \mathbf{1})^q + \sum_{|J|=n-3} \sum_{i=0,2} \sum_{j=0}^{2} ([p_2, 0, p_1] T_{ij} T_J \mathbf{1})^q \\
&= s_1 u_{n-2} + \Big(\sum_{|J|=n-3} \sum_{i=0,2} ([p_2, p_1] \bar{T}_i \begin{bmatrix} p_0 \\ p_3 \end{bmatrix} \mathbf{e}_1^t T_J \mathbf{1})^q + \sum_{|J|=n-3} \sum_{i,j=0,2} ([p_2, 0, p_1] T_{ij} T_J \mathbf{1})^q \Big) \\
&= s_1 u_{n-2} + s_2 u_{n-3} + \sum_{|J|=n-3} \sum_{i,j=0,2} ([p_2, 0, p_1] T_{ij} T_J \mathbf{1})^q.
\end{aligned}$$

By repeating the above argument, we have

$$I_n^{(1)} = \sum_{k=1}^{n-1} s_k u_{n-1-k} + \sum_{J \in \{0,2\}^{n-1}} ([p_2, 0, p_1] T_J \mathbf{1})^q.$$

From which statement (i) follows. To prove (ii), we let $f_k = r^{k+1} s_k$ and $g_k = r^k u_k$. Then we can rewrite (i) as

$$g_n = \sum_{k=0}^{n-1} f_k g_{n-1-k} + r^n b_n$$

with $\sum_{k=0}^{\infty} f_k = 1$. This is a renewal equation; if $\sum_{k=1}^{\infty} k f_k < \infty$ and $\sum_{k=1}^{\infty} r^k b_k < \infty$, then by the Renewal Theorem [Fe, p.330, Theorem 1], we have

$$\lim_{n\to\infty} g_n = \lim_{n\to\infty} r^n u_n = c$$

for some constant $c > 0$. This implies that $\lim_{n\to\infty} (u_n)^{1/n} = r^{-1}$. □

We remark that if $p_i > 0, \ i = 0, 1, 2, 3$ and $q > 0$, then the sequence $\{s_n\}_{n=0}^{\infty}$ is sub-multiplicative, i.e., there exists a constant $c > 0$ such that $s_{n+1} s_{m+1} \geq c s_{n+m+1}$.

This is sufficient for the existence of $r = r(q)$, $q > 0$, satisfying (2.7) (see the proof of Lemma 3.3 and Proposition 3.4 for detail).

THEOREM 2.5. *Suppose $p_i > 0,\ i = 0, 1, 2, 3$ and assume that $r(q)$ satisfies (2.7). Then the L^q-spectrum $\tau(q)$ of μ defined by (2.4) is given by*

$$\tau(q) = \min\{-q\log_3 p_0,\ -q\log_3 p_3,\ \log_3 r(q)\}, \quad q \in \mathbb{R}$$

and $\tau(q) = \log_3 r(q)$ if $q > 0$.

Proof. In addition to the above u_n, we define

$$t_n := t_n(q) = \sum_{|J|=n} (\mathbf{e}_0^t T_J \mathbf{1})^q, \quad v_n := v_n(q) = \sum_{|J|=n} (\mathbf{e}_2^t T_J \mathbf{1})^q,$$

then

$$\begin{aligned} t_n &= \sum_{|J|=n-1} \left[([p_0, 0, 0]T_J\mathbf{1})^q + ([0, p_0, 0]T_J\mathbf{1})^q + ([p_1, 0, p_0]T_J\mathbf{1})^q\right] \\ &= p_0^q t_{n-1} + p_0^q u_{n-1} + \sum_{|J|=n-1} \big([p_1, 0, p_0]T_J\mathbf{1}\big)^q. \end{aligned}$$

Note that $p_i > 0,\ i = 0, 1, 2, 3$, hence there exists $C > 0$ such that

$$\sum_{|J|=n-1} \big([p_1, 0, p_0]T_J\mathbf{1}\big)^q \ \le\ C \sum_{|J|=n-1} \big([p_2, 0, p_1]T_J\mathbf{1}\big)^q \ =\ C I_n^{(1)} < C u_n,$$

Therefore

$$\begin{aligned} t_n &\le p_0^q t_{n-1} + p_0^q u_{n-1} + C u_n \\ &\ \vdots \\ &\le 2p_0^{nq} + (C+1)\sum_{k=1}^{n-1} p_0^{kq} u_{n-k} + C u_n \\ &< (C+1)\sum_{k=0}^{n} p_0^{kq} u_{n-k}. \end{aligned}$$

Similarly we have $v_n < (1 + C)\sum_{k=0}^{n} p_3^{kq} u_{n-k}$. Hence

$$\sum_{i=0}^{2} \sum_{|J|=n} (\mathbf{e}_i^t T_J \mathbf{1})^q = t_n + u_n + v_n < u_n + (1 + C)\sum_{k=0}^{n} (p_0^{kq} + p_3^{kq}) u_{n-k}.$$

Recall the elementary identities: If $x_n > 0$, $\ y_n > 0$ and $\lim_{n\to\infty}(x_n)^{1/n} = x$, $\lim_{n\to\infty}(y_n)^{1/n} = y$, then

$$\lim_{n\to\infty} (x_n + y_n)^{1/n} = \lim_{n\to\infty} \Big(\sum_{k=1}^{n} x_k y_{n-k}\Big)^{1/n} = \max\{x,\ y\}. \tag{2.8}$$

By using this and $\lim_{n\to\infty} u_n^{-1/n} = r(q)^{-1}$ (Lemma 2.4 (ii)), we have

$$\limsup_{n\to\infty} \Big(\sum_{i=0}^{2} \sum_{|J|=n} (\mathbf{e}_i^t T_J \mathbf{1})^q\Big)^{1/n} \le \max\left\{p_0^q,\ p_3^q,\ r(q)^{-1}\right\}. \tag{2.9}$$

On the other hand, observe that

$$(\mathbf{e}_0^t T_0^n \mathbf{1})^q + (\mathbf{e}_2^t T_2^n \mathbf{1})^q + u_n = p_0^{nq} + p_3^{nq} + u_n < \sum_{i=0}^{2} \sum_{|J|=n} (\mathbf{e}_i^t T_J \mathbf{1})^q.$$

Hence

$$\max\{p_0^q,\ p_3^q,\ r(q)^{-1}\} \leq \liminf_{n\to\infty} \big(\sum_{i=0}^{2} \sum_{|J|=n} (\mathbf{e}_i^t T_J \mathbf{1})^q\big)^{1/n}.$$

This together with (2.9) prove that

$$\lim_{n\to\infty} \big(\sum_{i=0}^{2} \sum_{|J|=n} (\mathbf{e}_i^t T_J \mathbf{1})^q\big)^{1/n} = \max\{p_0^q,\ p_3^q,\ r(q)^{-1}\},$$

hence yields the first part of the theorem.

To prove the second part, we let s_n be as in the last lemma, then for $q > 0$,

$$s_n = \sum_{J \in \{0,2\}^{n-1}} \Big([p_2, p_1]\bar{T}_J \begin{bmatrix} p_0 \\ p_3 \end{bmatrix}\Big)^q > \Big([p_2, p_1]\bar{T}_0^{n-1} \begin{bmatrix} p_0 \\ p_3 \end{bmatrix}\Big)^q > \big(p_2 p_0^n\big)^q.$$

We claim that $-q \log_3 p_0 \geq \log_3 r(q)$, equivalently, $r(q) \leq p_0^{-q}$. For otherwise we have $r = r(q) > p_0^{-q}$, it implies

$$s_n r^n \ > \ (p_2 p_0^n)^q p_0^{-nq} \ = \ p_2^q > 0.$$

It follows that $\sum_{n=0}^{\infty} s_n r^{n+1} = \infty$ which contradicts the assumption on $r := r(q)$. This proves the claim. Similarly we can prove $-q \log_3 p_3 \geq \log_3 r(q)$ and the second part of the theorem follows. □

By the theorem, we see that $\tau(q)$ may have non-differentiable points; this actually happens in the case of 3-fold convolution of the Cantor measure, to be seen in Theorem 3.6. In the following we prove a result on the restriction of the measure μ to the interior of the support $[0, 3]$. This will eliminate the non-differentiable point and will be used in Part II to set up the modified multifractal formalism. For each integer $m \geq 0$, let $\mu_m := \mu|_{[3^{-m}, 3-3^{-m}]}$, the restriction of μ on the interval $[3^{-m}, 3 - 3^{-m}]$. Let $\tau_m(q)$ be the L^q-spectrum of μ_m, i.e.,

$$\tau_m(q) = -\lim_{n\to\infty} \frac{1}{n} \log_3 \Big(\sum_{i=0}^{2} \sum_{|J|=n} \mu_m([\![J]\!] + i)^q\Big).$$

Proposition 2.6. *Suppose $p_i > 0$ for $i = 0, 1, 2, 3$ and suppose $r(q)$ satisfies (2.7). Then for any $m \geq 0$ and $q \in \mathbb{R}$, $\tau_m(q) = \log_3 r(q)$ is independent of m.*

Proof. For $n > m$, we let

$$A_n^{(i)} = \sum_{|J|=n} \mu_m([\![J]\!] + i)^q, \quad i = 0, 1, 2.$$

For $i = 1$, it is clear that $\mu_m([\![J]\!]+1) = \mu([\![J]\!]+1)$ and by (2.5), we have

$$A_n^{(1)} = \sum_{|J|=n} \mu([\![J]\!]+1)^q \approx \sum_{|J|=n} \left(\mathbf{e}_1^t T_J \mathbf{1}\right)^q = u_n.$$

For $i = 0$,

$$\begin{aligned} A_n^{(0)} &= \sum_{k=1}^{m} \sum_{|J|=n-k} \left(\mu([\![0^{k-1}1J]\!])^q + \mu([\![0^{k-1}2J]\!])^q\right) \\ &\approx \sum_{k=1}^{m} \sum_{|J|=n-k} \left(\left(\mathbf{e}_0^t T_0^{k-1} T_1 T_J \mathbf{1}\right)^q + \left(\mathbf{e}_0^t T_0^{k-1} T_2 T_J \mathbf{1}\right)^q \right) \\ &\approx \sum_{k=1}^{m} \sum_{|J|=n-k} \left((\mathbf{e}_1^t T_J \mathbf{1})^q + ([p_1, 0, p_0] T_J \mathbf{1})^q\right) \\ &\approx \sum_{k=1}^{m} \left(u_{n-k} + I_{n-k+1}^{(1)}\right) \end{aligned}$$

(the second $\approx$ is by (2.5) with the associated constant depends on m). From the proof of Lemma 2.4, there exists a constant $C > 0$ such that

$$Cu_{n-2} \le I_n^{(1)} < u_n.$$

Hence $\lim_{n\to\infty} \left(I_n^{(1)}\right)^{1/n} = \lim_{n\to\infty} (u_n)^{1/n} = r^{-1}$. This together with (2.8) proves that $\lim_{n\to\infty} \left(A_n^{(0)}\right)^{1/n} = r(q)^{-1}$. Similarly, we have $\lim_{n\to\infty} \left(A_n^{(2)}\right)^{1/n} = r(q)^{-1}$. By making use of (2.8) once more, we have

$$\lim_{n\to\infty} \Big(\sum_{i=0}^{2} \sum_{|J|=n} \mu_m([\![J]\!]+i)^q\Big)^{1/n} = r(q)^{-1},$$

which concludes the proposition. $\square$

3. 3-fold convolution of the Cantor measure. The 3-fold convolution of the Cantor measure μ is the self-similar measure defined by (2.1) with weight $[p_0, p_1, p_2, p_3] = \frac{1}{8}[1, 3, 3, 1]$. This is the special case in the last section. Our task is to simplify the previous expression of s_n and to justify condition (2.7). Let

$$M_0 = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}.$$

Define $\tilde{s}_0 := \tilde{s}_0(q) = 2$ and

$$\tilde{s}_n := \tilde{s}_n(q) = \sum_{J\in\{0,2\}^{n-1}} (\mathbf{1}^t M_J \mathbf{1})^q, \quad n \ge 1,$$

and $\tilde{r}(q) > 0$ satisfies $3^q \sum_{k=0}^{\infty} \tilde{s}_k \tilde{r}(q)^{k+1} = 1$.

LEMMA 3.1. *With the above notations, we have*

$$s_n = 3^q 8^{-(n+1)q} \tilde{s}_n, \quad n \ge 0 \quad \textit{and} \quad r(q) = 8^q \tilde{r}(q)$$

where s_n and $r(q)$ are defined as in the last section.

Proof. A direct calculation shows that for any $J \in \{0,2\}^{n-1}$,

$$[p_2, p_1]\bar{T}_J \begin{bmatrix} p_0 \\ p_3 \end{bmatrix} = \frac{3}{8^{n+1}}\mathbf{1}^t M_J \mathbf{1}.$$

Hence $s_n = 3^q 8^{-(n+1)q}\tilde{s}_n$, and the last part of the lemma follows from $\sum_{k=0}^{\infty} s_k r^{k+1} = 1$. ∎

We need the product M_J for two special indices: $J_n^0 = \varepsilon \cdots 2020$, $J_n^2 = (2-\varepsilon)\cdots 0202$ of alternative sequences of 0 and 2 with length n, where $\varepsilon = 0$ if n is odd and $\varepsilon = 2$ if n is even.

LEMMA 3.2. *For any $J \in \{0,2\}^n$, we write $\|M_J\| = \mathbf{1}^t M_J \mathbf{1}$ and $[a_J, b_J] := \mathbf{1}^t M_J$. Then*

(i) $a_{J_n^0} = b_{J_n^2}$, $a_{J_n^2} = b_{J_n^0}$, $\|M_{J_n^0}\| = \|M_{J_n^2}\|$;

(ii) $a_{J_n^0} < b_{J_n^0}$;

(iii) $2\sqrt{\lambda_1^n} < \|M_{J_n^0}\| < 5\sqrt{\lambda_1^n}$, *where* $\lambda_1 = \frac{7+\sqrt{13}}{2}$;

(iv) $\|M_{J_n^0}\| = \min\{\|M_J\| : |J| = n\}$ *and* $a_{J_n^0} = \min\{a_J, b_J : |J| = n\}$.

Proof. (i) Let $Q = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, then $Q^{-1} = Q$ and $M_{J_n^0} = Q^{-1}M_{J_n^2}Q$. It follows that $\|M_{J_n^0}\| = \|M_{J_n^2}\|$, $a_{J_n^0} = b_{J_n^2}$ and $a_{J_n^2} = b_{J_n^0}$.
(ii) For $n = 1$, the statement is trivial. In the following, we assume that $n > 1$. Let $\lambda_1 = \frac{7+\sqrt{13}}{2}$, $\lambda_2 = \frac{7-\sqrt{13}}{2}$ be the eigenvalues of M_2M_0. Let $P = \left[\begin{smallmatrix} \frac{1+\sqrt{13}}{2} & \frac{1-\sqrt{13}}{2} \\ 1 & 1 \end{smallmatrix}\right]$, then $P^{-1} = \frac{1}{\sqrt{13}}\left[\begin{smallmatrix} 1 & \frac{\sqrt{13}-1}{2} \\ -1 & \frac{\sqrt{13}+1}{2} \end{smallmatrix}\right]$ and $P^{-1}(M_2M_0)P = \left[\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right]$. By a direct calculation, we have

$$[a_{J_{2n}^0},\ b_{J_{2n}^0}] = \frac{1}{2\sqrt{13}}[(\sqrt{13}+3)\lambda_1^n + (\sqrt{13}-3)\lambda_2^n,\ \ (\sqrt{13}+5)\lambda_1^n + (\sqrt{13}-5)\lambda_2^n],$$

and by (i),

$$[a_{J_{2n+1}^0},\ b_{J_{2n+1}^0}] \ = \ [a_{J_{2n}^2},\ b_{J_{2n}^2}]M_0 \ = \ [a_{J_{2n}^0} + b_{J_{2n}^0},\ 3a_{J_{2n}^0}].$$

Hence (ii) follows from

$$b_{J_{2n}^0} - a_{J_{2n}^0} = \frac{1}{\sqrt{13}}(\lambda_1^n - \lambda_2^n) > 0$$

and

$$b_{J_{2n+1}^0} - a_{J_{2n+1}^0} = \frac{1}{2\sqrt{13}}[(\sqrt{13}+1)\lambda_1^n + (\sqrt{13}-1)\lambda_2^n] > 0.$$

(iii) We need only observe that

$$\|M_{J_{2n}^0}\| = \frac{1}{\sqrt{13}}[(\sqrt{13}+4)\lambda_1^n + (\sqrt{13}-4)\lambda_2^n]$$

and

$$\|M_{J_{2n+1}^0}\| = \frac{1}{2\sqrt{13}}[(5\sqrt{13}+17)\lambda_1^n + (5\sqrt{13}-17)\lambda_2^n].$$

(iv) We see that the minimal identity is trivial for $n = 1$. Suppose that the statements are true for n, then for any $|J| = n$, we have

$$\|M_{J0}\| = a_J + 4b_J = \|M_J\| + 3b_J \geq \|M_{J_n^2}\| + 3b_{J_n^2} = \|M_{J_n^2 0}\| = \|M_{J_{n+1}^0}\|.$$

Similarly, we have $\|M_{J2}\| \geq \|M_{J_{n+1}^2}\|$. Hence the first identity follows. For the second identity, we note that

$$[a_{J0}, b_{J0}] = [a_J + b_J, 3b_J] = [\|M_J\|, 3b_J] \geq [\|M_{J_n^2}\|, 3b_{J_n^2}] = [a_{J_{n+1}^0}, b_{J_{n+1}^0}].$$

Similarly $[a_{J2}, b_{J2}] \geq [a_{J_{n+1}^2}, b_{J_{n+1}^2}]$. The conclusion follows immediately from statements (i) and (ii). □

LEMMA 3.3. *Let $m, n \geq 0$. Then*
(i) *If $q \geq 0$, then $\tilde{s}_{m+1}\tilde{s}_{n+1} \geq \tilde{s}_{m+n+1}$ and $\lim_{n\to\infty}(\tilde{s}_{n+1})^{1/n} = \inf_{n\geq 1}(\tilde{s}_{n+1})^{1/n}$.*
(ii) *If $q < 0$, then for all integer $n \geq 0$, we have*

$$2^q\big(2\big(\frac{5}{2}\big)^q\big)^n \leq \tilde{s}_{n+1} \leq 2^q\big(2\sqrt{\lambda_1^q}\big)^n. \tag{3.1}$$

Proof. For $q \geq 0$, note that

$$\tilde{s}_{m+1}\tilde{s}_{n+1} = \sum_{|J|=m} \|M_J\|^q \sum_{|J'|=n} \|M_{J'}\|^q = \sum_{|J|=m}\sum_{|J'|=n} \big(\mathbf{1}^t M_J \mathbf{1}\mathbf{1}^t M_{J'}\mathbf{1}\big)^q.$$

Observe that $\mathbf{1}\mathbf{1}^t = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, hence $\mathbf{1}^t M_J \mathbf{1}\mathbf{1}^t M_{J'}\mathbf{1} \geq \|M_{JJ'}\|$. This implies $\tilde{s}_{m+1}\tilde{s}_{n+1} \geq \tilde{s}_{m+n+1}$; the expression of the limit follows from this (see [Fa, Corollary1.3]).

For (ii), it is easy to check that for any $J \in \{0, 2\}^n$, $\|M_{0J}\| + \|M_{2J}\| = 5\|M_J\|$. Note that $\tilde{s}_1 = 2^q$ and by making use of the convexity of x^q $(q < 0)$, we have

$$\tilde{s}_{n+1} = \sum_{|J|=n-1} \big(\|M_{0J}\|^q + \|M_{2J}\|^q\big) \geq 2\big(\frac{5}{2}\big)^q \sum_{|J|=n-1} \|M_J\|^q = 2\big(\frac{5}{2}\big)^q \tilde{s}_n.$$

The first inequality of (3.1) follows from this. Lemma 3.2 (iii), (iv) imply the second inequality. □

Now we prove the existence of $r(q)$ as in Lemma 2.4. In view of Lemma 3.1, it suffices to consider $\tilde{r}(q)$. Let

$$F(q, x) = 3^q \sum_{n=0}^{\infty} \tilde{s}_n(q) x^{n+1}.$$

PROPOSITION 3.4. *For any $q \geq -2$, there exists a unique $\tilde{r}(q) > 0$ such that $F(q, \tilde{r}(q)) = 1$. Furthermore $\tilde{r}(q)$ is real analytic for $q \geq -2$.*

Proof. For any fixed $q \in \mathbb{R}$, let $R(q)$ denote the radius of convergence of the power series $\sum_{n=0}^{\infty} \tilde{s}_n(q)x^{n+1}$. We first show that $F(q, R(q)) > 1$:

Case (i): If $q \geq 0$, then by Lemma 3.3 (i),

$$R(q) = \lim_{n\to\infty}(\tilde{s}_{n+1})^{-1/n} = \sup_{n\geq 1}(\tilde{s}_{n+1})^{-1/n} \geq (\tilde{s}_{k+1})^{-1/k} \quad \text{for all } k > 0.$$

Hence $\tilde{s}_{k+1}R^k(q) \geq 1$ and it follows that $F(q, R(q)) = \infty$.

Case (ii): For $-2 \leq q < 0$, Lemma 3.3 (ii) implies that

$$R(q) = \lim_{n\to\infty} (\tilde{s}_{n+1})^{-1/n} \ \geq \ \frac{1}{2}\sqrt{\lambda_1^{-q}} \ = \ \frac{1}{2}\Big(\frac{2}{1+\sqrt{13}}\Big)^q := \frac{A^q}{2}.$$

Making use of Lemma 3.3 once more, we have

$$\begin{aligned} F(q, R(q)) &= 3^q \sum_{n=0}^{\infty} \tilde{s}_n R^{n+1}(q) \\ &\geq 3^q \Big(A^q + \sum_{n=1}^{\infty} 2^q \big(2(\frac{5}{2})^q\big)^{n-1} (\frac{A^q}{2})^{n+1} \Big) \\ &= (3A)^q + \frac{(6A^2)^q}{4} \Big(1 - (\frac{5A}{2})^q\Big)^{-1} \\ &\geq (3A)^{-2} + \frac{(6A^2)^{-2}}{4} \Big(1 - (\frac{5A}{2})^{-2}\Big)^{-1} \approx 1.8 > 1 \end{aligned}$$

(the last $\geq$ follows from a direct check on the increasingness of the expression for $-2 \leq q < 0$). Note that $F(q, 0) = 0$ and for any fixed q, $F(q, x)$ is an increasing function of x, hence there exists a unique $0 < \tilde{r}(q) < R(q)$ such that $F(q, \tilde{r}(q)) = 1$ ($q \geq -2$). The analyticity of $\tilde{r}(q)$ follows from the power series expression. ☐

We remark that the above $\tilde{r}(q)$ actually exists for all $q \in \mathbb{R}$, but it needs more work and will be proved in [FLW].

Proposition 3.5. *There exists a unique* $q_0 \in (-1.14996,\ -1.14960)$ *such that* $\tilde{r}(q_0) = 1$.

Proof. If $q < q_1 := \min\big\{-\log 2/\log\frac{5}{2},\ -\log 4/\log\lambda_1\big\} \approx -0.756$, we let

$$f_N(q) = 3^q\Big(\sum_{k=0}^{N} \tilde{s}_k \ + \ 2^q \frac{\big(2(\frac{5}{2})^q\big)^N}{1 - 2(\frac{5}{2})^q}\Big), \quad g_N(q) = 3^q\Big(\sum_{k=0}^{N} \tilde{s}_k \ + \ 2^q \frac{\big(2\sqrt{\lambda_1^q}\big)^N}{1 - 2\sqrt{\lambda_1^q}}\Big).$$

Then by Lemma 3.3 (ii),

$$f_N(q) \leq F(q, 1) \leq g_N(q)$$

holds for all $N \in \mathbb{N}$ and $q < q_1$. A direct calculation shows that $f_{24}(-1.14960) = 1.0000002 > 1$ and $g_{24}(-1.14996) = 0.999981 < 1$. The continuity of $F(q, 1)$ implies that there exists $q_0 \in (-1.14996,\ -1.14960)$ such that $\tilde{r}(q_0) = 1$. The uniqueness follows from the monotonicity of $\tilde{r}(q)$. ☐

Our main theorem is

Theorem 3.6. *Let* μ *be the 3-fold convolution of the Cantor measure. Then the* L^q*-spectrum* $\tau(q)$ *of* μ *is given by*

$$\tau(q) = \begin{cases} q\log_3 8 + \log_3 \tilde{r}(q) & \text{if} \quad q \geq q_0, \\ q\log_3 8 & \text{if} \quad q < q_0, \end{cases} \tag{3.2}$$

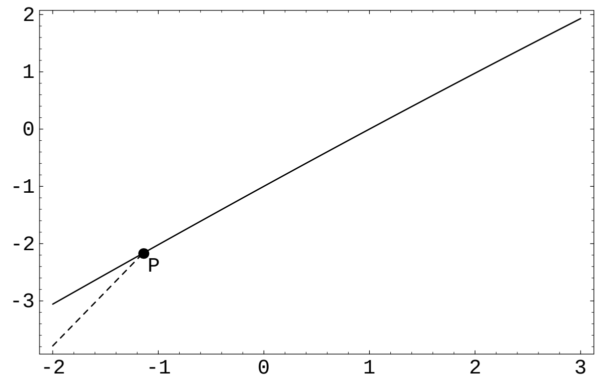

FIG. 3.1. *The curve of $\tau(q)$ and $\tilde{\tau}(q)$. $\tilde{\tau}(q)$ is the solid curve. If $q \geq q_0$, then $\tau(q) = \tilde{\tau}(q)$; if $q < q_0$, $\tau(q)$ is straight line, the dotted line.*

where q_0 is defined by Proposition 3.5 and $\tau(q)$ is differentiable except at q_0.

Proof. By Proposition 3.4, we see that $r(q) = 8^q \tilde{r}(q)$ satisfies the conditions in Theorem 2.5. Hence $\tau(q) = \min\{q \log_3 8,\ \log_3 r(q)\}$. Since $\log_3 r(q_0) = q_0 \log_3 8 + \log_3 \tilde{r}(q_0) = q_0 \log_3 8$ and $\tilde{r}(q)$ is a decreasing function of q (by (3.2)), it follows that $\tau(q) = \log_3 r(q)$ for $q > q_0$ and identity (3.2) follows. Since $\tilde{r}'(q_0) < 0$, we have $\tau'(q_0+) < \tau'(q_0-)$. This implies that $\tau(q)$ is not differentiable at q_0. ☐

REMARK. For $q > 0$, Lau and Ngai [LN3] gave the same formula of $\tau(q)$ by using a different method. If q is an integer they reduced $F(q,x) = 1$ to a polynomial equation, hence $\tau(q)$ can be calculated easily for such q. Furthermore they gave an explicit expression of $\tau'(1)$ (≈ 0.9884). (Recall that $\tau'(1)$ is the Hausdorff and entropy dimensions of μ under certain condition [N]).

4. Other examples. Let $\tilde{S}_j(x) = \lambda(x+j), j = 0,1,3$, with $0 < \lambda < 1$, and let $K(\lambda)$ be the self-similar set of the IFS. Keane *et al* [KSS] asked the following question: What is the Hausdorff dimension of $K(\lambda)$? They call this the (0,1,3)-problem. Pollicott and Simon gave some partial results on the question [PoS]. If $\lambda = \frac{1}{3}$, then this system is equivalent to the IFS $S_j(x) = \frac{1}{3}(x+2j), j = 0,1,3$. Here we give a brief consideration of the self-similar measure μ generated by

$$S_j(x) = \frac{1}{3}(x+2j),\ j = 0,1,3 \quad \text{with weights} \quad p_0 = p_1 = p_3 = \frac{1}{3}. \tag{4.1}$$

For convenience, we denote $p_2 = 0$ and $S_2(x) = \frac{1}{3}(x+4)$. Then we can make use of the results in Section 2. Let

$$M_0 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},\quad M_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

and let $\tilde{s}_0 = \tilde{s}_0(q) = 1$,

$$\tilde{s}_n := \tilde{s}_n(q) = \sum_{J \in \{0,2\}^{n-1}} \left([0,1] M_J \mathbf{1}\right)^q, \quad n \geq 1.$$

PROPOSITION 4.1. *There exists a unique real analytic function $\tilde{r}(q) > 0$, $q \in \mathbb{R}$ satisfies $\sum_{k=0}^{\infty} \tilde{s}_k(q) \tilde{r}^{k+1}(q) = 1$.*

Proof. Observe that

$$\tilde{s}_n = \sum_{|J|=n-2} \big(([0,1]M_0M_J\mathbf{1})^q + ([0,1]M_2M_J\mathbf{1})^q\big) = \tilde{s}_{n-1} + \sum_{|J|=n-2} ([1,0]M_J\mathbf{1})^q.$$

Let $w_0 = 1$, $w_n := w_n(q) = \sum_{|J|=n-1}([1,0]M_J\mathbf{1})^q,\ n \geq 1$. Then

$$w_{n+1} \geq \sum_{|J|=n-1} ([1,0]M_2M_J\mathbf{1})^q = \sum_{|J|=n-1} ([1,1]M_J\mathbf{1})^q \geq \tilde{s}_n \geq w_{n-1}, \tag{4.2}$$

for all $q \geq 0$. Also a direct calculation yields

$$w_n = \sum_{k=1}^{n} k^q w_{n-k}.$$

Note that for each $q > 0$, there exists ρ (depends on q) such that $\sum_{k=1}^{\infty} k^q\rho^k = 1$. The renewal theorem hence implies

$$\lim_{n\to\infty} \rho^n w_n = c > 0 \tag{4.3}$$

and it follows that $\lim_{n\to\infty}(\tilde{s}_n)^{1/n} = \lim_{n\to\infty}(w_n)^{1/n} = \rho^{-1}$. Let $R(q)$ be the radius of convergence of the series $F(q,x) = \sum_{k=0}^{\infty} \tilde{s}_k x^{k+1}$, then $R(q) = \lim_{n\to\infty}(w_n)^{-1/n} = \rho$. (4.2) and (4.3) imply that $F(q, R(q)) = +\infty$ for all $q \geq 0$. Hence there exists $\tilde{r}(q) > 0$ such that $F(q, \tilde{r}(q)) = 1$.

For $q < 0$, we observe that

$$\mathbf{1}[0,1]M_0 = M_0 \quad \text{and} \quad \mathbf{1}[0,1]M_2 = \mathbf{1}[0,1] \leq M_2,$$

(the notation $[a_{ij}]_{2\times 2} \leq [b_{ij}]_{2\times 2}$ means $a_{ij} \leq b_{ij},\ i,j = 1,2$). Hence

$$\tilde{s}_{n+1}\tilde{s}_{m+1} = \sum_{|J|=n}\sum_{|J'|=m} \big([0,1]M_J\mathbf{1}[0,1]M_{J'}\mathbf{1}\big)^q \geq \tilde{s}_{n+m+1}.$$

A similar argument as in the first part of the proof of Proposition 3.4 yields the existence of $\tilde{r}(q)$. Since $(q, \tilde{r}(q))$ is an interior point of $D = \{(q,x) :\ F(q,x) < \infty\}$, the implicit function theorem implies the real analyticity of $\tilde{r}(q)$. ∎

Let μ be the self-similar measure defined by (4.1), then we have

THEOREM 4.2. *The L^q-spectrum of μ is given by the real analytic function*

$$\tau(q) = q + \log_3 \tilde{r}(q), \quad q \in \mathbb{R}. \tag{4.4}$$

Proof. We adopt the notations in Section 2 for this special case $p_0 = p_1 = p_3 = \frac{1}{3},\ p_2 = 0$. Hence

$$T_0 = \frac{1}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad T_1 = \frac{1}{3}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad T_2 = \frac{1}{3}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

A direct calculation yields:

$$
\begin{aligned}
u_n &:= \sum_{|J|=n} (\mathbf{e}_1^t T_J \mathbf{1})^q = 3^{-q}(u_{n-1} + v_{n-1}), \\
v_n &:= \sum_{|J|=n} (\mathbf{e}_2^t T_J \mathbf{1})^q = 3^{-q}(t_{n-1} + u_{n-1} + v_{n-1}), \\
t_n &:= \sum_{|J|=n} (\mathbf{e}_0^t T_J \mathbf{1})^q = 3^{-q}\big(t_{n-1} + u_{n-1} + \sum_{|J|=n-1} (\mathbf{e}_3^t T_J \mathbf{1})^q\big),
\end{aligned}
$$

where $\mathbf{e}_3^t = [1, 0, 1]$. This implies that

$$u_n \geq 3^{-q} v_{n-1} \geq 3^{-2q} t_{n-2} \geq 3^{-3q} u_{n-3}.$$

By Lemma 2.4 (ii), we have

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} (v_n)^{1/n} = \lim_{n\to\infty} (t_n)^{1/n} = r^{-1}, \tag{4.5}$$

where $r = r(q)$ is as in Lemma 2.4. It follows from the definition of $\tau(q)$, (2.8) and (4.5) that

$$\tau(q) = -\lim_{n\to\infty} \frac{1}{n} \log_3(t_n + u_n + v_n) = \log_3 r(q).$$

A similar argument as in Lemma 3.1 shows that $s_n = 3^{-(n+1)q}\tilde{s}_n$ and $r(q) = 3^q \tilde{r}(q)$. Hence (4.4) follows. ☐

Recall that the entropy dimension of μ is given by $\tau'(1)$.

Corollary 4.3. *Let $c_J = [0, 1] M_J \mathbf{1}$. Then*

$$\tau'(1) = 1 - \frac{1}{54 \log 3} \sum_{k=0}^{\infty} 3^{-k} \sum_{|J|=k} c_J \log(c_J) \ (\approx 0.8517). \tag{4.6}$$

Proof. By taking derivative on both side of $\sum_{k=0}^{\infty} \tilde{s}_k(q)\tilde{r}(q)^{k+1} = 1$. We have

$$\tilde{r}'(q) = -\frac{\sum_{k=0}^{\infty} \tilde{s}_k'(q)\tilde{r}(q)^{k+1}}{\sum_{k=0}^{\infty} (k+1)\tilde{s}_k(q)\tilde{r}(q)^k}. \tag{4.7}$$

To evaluate each term in the expression, we first observe that $\tilde{s}_k'(1) = \sum_{|J|=k-1} c_J \log c_J$. Since $\tau(1) = 0$, we conclude from (4.4) that $\tilde{r}(1) = \frac{1}{3}$. Next we note that

$$\tilde{s}_k(1) = \sum_{|J|=k-1} [0, 1] M_J \mathbf{1} = [0, 1](M_0 + M_2)^{k-1}\mathbf{1} = \frac{\sqrt{5}+1}{2\sqrt{5}} \lambda_1^{k-1} + \frac{\sqrt{5}-1}{2\sqrt{5}} \lambda_2^{k-1},$$

where $k \geq 1$ and $\lambda_1 = \frac{3+\sqrt{5}}{2}$, $\lambda_2 = \frac{3-\sqrt{5}}{2}$ are the eigenvalues of $M_0 + M_2$. It follows from a direct calculation that

$$\sum_{k=0}^{\infty} (k+1)\tilde{s}_k(1)\tilde{r}(1)^k = 1 + \sum_{k=1}^{\infty} (k+1) \left(\frac{\sqrt{5}+1}{2\sqrt{5}} \lambda_1^{k-1} + \frac{\sqrt{5}-1}{2\sqrt{5}} \lambda_2^{k-1} \right) 3^{-k} = 18.$$

The corollary follows by (4.7). $\square$

We remark that in [HNW] it is proved that the range of the local dimensions of μ (i.e., the domain of the dimension spectrum $f(\alpha)$) is $[\frac{2}{3}, 1]$. We can prove a corresponding result for $\tau(q)$.

PROPOSITION 4.4. *Let μ be the self-similar measure defined by (4.1), then* (i) $\lim_{q\to-\infty} \frac{\tau(q)}{q} = 1$; (ii) $\lim_{q\to+\infty} \frac{\tau(q)}{q} = \frac{2}{3}$.

Proof. (i) If $q < 0$, then for any J, $[0,1]M_J\mathbf{1} \geq 1$, hence

$$\tilde{s}_n = \sum_{|J|=n-1} \big([0,1]M_J\mathbf{1}\big)^q \leq 2^{n-1}, \quad n \geq 1.$$

It follows that

$$1 = \sum_{n=0}^{\infty} \tilde{s}_n \tilde{r}^{n+1} \leq \tilde{r} + \sum_{n=1}^{\infty} 2^{n-1}\tilde{r}^{n+1} = \tilde{r} + \tilde{r}^2 \sum_{n=1}^{\infty} (2\tilde{r})^{n-1} = \tilde{r} + \frac{\tilde{r}^2}{1-2\tilde{r}}.$$

Hence $\frac{1}{3} < \tilde{r}$ (≤ 1). By making use of (4.4), (i) follows.

(ii) If $q \geq 0$, then from the proof of Proposition 4.1 we know that $\tilde{r} < \rho$, where ρ satisfies $\sum_{k=1}^{\infty} k^q\rho^k = 1$. This implies that $3^q\tilde{r}^3 \leq \sum_{k=1}^{\infty} k^q\rho^k = 1$, and by making use of (4.4) again, we have

$$\limsup_{q\to+\infty} \frac{\tau(q)}{q} \leq \frac{2}{3}.$$

Now we need to prove that $\liminf_{q\to+\infty} \frac{\tau(q)}{q} \geq \frac{2}{3}$. For this, we estimate $\tilde{s}_m$; we try to find an index $J = j_1 \cdots j_m$ such that $[0,1]M_J\mathbf{1}$ attains its maximum. We observe that the following procedures will not decrease the product:
(a) If the initial segment of J is 2^k0, we replace it by 02^k (since $[0,1]M_2^kM_0 \leq [0,1]M_0M_2^k$);
(b) If the last index $j_m = 0$, we replace it by $j_m = 2$ (since $M_0\mathbf{1} \leq M_2\mathbf{1}$);
(c) We replace the segment 0^k2 in J by 02^k (since $M_0^kM_2 \leq M_0M_2^k$).

From the above, we need only consider the index

$$J = 02^{k_1-1}02^{k_2-1}\cdots02^{k_s-1}, \quad k_j > 1, \quad \sum_j k_j = n.$$

Note that $M_0 = \mathbf{1}[1,0]$, a direct calculation shows that

$$[0,1]M_J\mathbf{1} = k_1k_2\cdots k_s.$$

We claim that $k_1k_2\cdots k_s \leq 3^{n/3}$ (to compare with the one in [HNW]: $k_1k_2\cdots k_s \leq 3^{[n/3]+1}$, where $[x]$ is the integer part of x, we adopt their method slightly to prove the claim). We observe that if $k_j \geq 4$ for some j, we can replace it by the larger value $2(k_j-2)$ in the product $k_1k_2\cdots k_s$, hence we may assume that $k_j = 2$ or $k_j = 3$ for all $j = 1, 2, \cdots, s$. Next, by replacing any three $k_j = 2$ by 3^2, we have $k_1k_2\cdots k_s \leq 3^l\cdot2^{l'}$, where $0 \le l' \le 2$ and $\sum_j k_j = n = 3l + 2l'$. It follows that $3^l \cdot 2^{l'} \le 3^{n/3}$ and the claim is proved. Hence, we have

$$\tilde{s}_{n+1} = \sum_{|J|=n} \big([0,1]M_J\mathbf{1}\big)^q \le 2^n \cdot 3^{nq/3}, \quad n \ge 0.$$

It follows that

$$1 = \sum_{n=0}^{\infty} \tilde{s}_n \tilde{r}^{n+1} \le \tilde{r} + \sum_{n=0}^{\infty} \big(2 \cdot 3^{q/3}\big)^n \tilde{r}^{n+2} = \tilde{r} + \frac{\tilde{r}^2}{1 - 2 \cdot 3^{q/3}\tilde{r}},$$

and hence $\tilde{r}(q) > 3^{-q/3-1}$. By making use of (4.4) one more time, the assertion follows. $\square$

To conclude this section we remark that the method in Section 2 can be applied to the more general cases as considered in [LN3] using another approach. Let μ be the self-similar measure defined by

$$S_j(x) = \frac{x}{N} + \frac{N-1}{N} j, \quad j = 0, 1, \cdots, N, \ (N \ge 3)$$

with probability weights $\{p_j\}_{j=0}^N$. Let

$$\bar{T}_0 = \begin{bmatrix} p_0 & 0 \\ p_N & p_{N-1} \end{bmatrix}, \ \ \bar{T}_{N-1} = \begin{bmatrix} p_{N-1} & p_N \\ 0 & p_0 \end{bmatrix}.$$

Let $s_0 = s_0(q) = \sum_{i=1}^{N-1} p_i^q$ and for $k \ge 1$, let

$$s_k = s_k(q) = \sum_{|J|=k-1} \sum_{i=1}^{N-2} \left([p_{i-1}, p_i] \bar{T}_J \begin{bmatrix} p_0 \\ p_N \end{bmatrix} \right)^q.$$

By a similar argument as in Section 2, we have

Theorem 4.5. *Suppose that $p_i > 0, i = 0, 1, \cdots, N$. Assume that for any $q \in \mathbb{R}$ there exists a unique positive $r = r(q)$ such that $\sum_{k=0}^{\infty} s_k r^{k+1} = 1$ and $\sum_{k=0}^{\infty} k s_k r^{k+1} < \infty$. Then* (i) *The L^q-spectrum of μ is given by*

$$\tau(q) = \min\{-q \log_N p_0, \ -q \log_N p_N, \ \log_N r(q)\}$$

and $\tau(q) = \log_N r(q)$ if $q > 0$; also

(ii) *The L^q-spectrum of μ_m, the restriction of μ on the interval $[N^{-m}, N - N^{-m}]$ is given by*

$$\tau_m(q) = \log_N r(q), \quad q \in \mathbb{R}$$

(which is independent of m).

By using the sub-multiplicative argument for the sequence $\{s_n\}_{n=0}^{\infty}$, it is not hard to show that $r(q)$ in the above theorem exists for $q \ge 0$ (see also [FL] for the more general case), but for $q < 0$, there are difficulties as is seen in Section 3 and we do not have an answer in general.

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