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WAVELET DECOMPOSITION OF CALDERÓN-ZYGMUND OPERATORS ON FUNCTION SPACES

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Abstract

We make use of the Beylkin-Coifman-Rokhlin wavelet decomposition algorithm on the Calderón-Zygmund kernel to obtain some fine estimates on the operator and prove the $T(1)$ theorem on Besov and Triebel-Lizorkin spaces. This extends previous results of Frazier *et al.*, and Han and Hofmann.

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1. Introduction

In recent years, there has been significant progress on the problem of boundedness of generalized Calderón-Zygmund operators on various function spaces. The operators in question can be described as follows. Let $K(x, y)$ be a continuous function defined on $(\mathbb{R}^n \times \mathbb{R}^n)\backslash\{x = y\}$ and let $T : \mathscr{D} \longrightarrow \mathscr{D}'$ be the linear operator associated with the kernel $K(x, y)$, that is,

$$\langle T\varphi, \psi\rangle = \iint_{\mathbb{R}^n\times\mathbb{R}^n} K(x, y)\varphi(y)\psi(x)\,dy\,dx$$

where $\varphi, \psi \in \mathscr{D}$ are C^∞-test functions on $\mathbb{R}^n$ with disjoint supports. For convenience, we write

$$\Delta K(x, x'; y, y') = \left|K(x, y) - K(x', y')\right| + \left|K(y, x) - K(y', x')\right|.$$

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It is customary to assume that $K(x, y)$ satisfies the following pointwise conditions:

$$|K(x, y)| \le C|x - y|^{-n}, \quad \text{and} \tag{1.1}$$

$$|\Delta K(x, x'; y, y)| \le C|x' - x|^{\gamma}|x - y|^{-n-\gamma} \quad \text{for } |x - y| \ge 2|x' - x|, \tag{1.2}$$

where $0 < \gamma \le 1$.

In their celebrated paper [4], David and Journé characterized the type of kernel $K(x, y)$ for which T is a bounded operator on L^2. This is now called the $T(1)$ theorem. They proved that under conditions (1.1) and (1.2) on $K(x, y)$, T extends to a bounded operator on L^2 if and only if both $T(1)$ and $T^*(1)$ are BMO functions, and T has the following *weak boundedness property* (WBP): For φ, $\psi \in \mathscr{D}$ with $\operatorname{diam}(\operatorname{supp}\varphi)$, $\operatorname{diam}(\operatorname{supp}\psi) \le t$,

$$|\langle T\varphi, \psi\rangle| \le t^n\big(\|\varphi\|_\infty + t\|\nabla\varphi\|_\infty\big)\big(\|\psi\|_\infty + t\|\nabla\psi\|_\infty\big). \tag{1.3}$$

Later, Meyer [11] improved the theorem by replacing the pointwise assumption with the following integral assumption on $K(x, y)$:

$$\sup_{r>0}\int_{r\le|x-y|<2r} \big(|K(x, y)| + |K(y, x)|\big)\, dy \le C, \quad \text{and} \tag{1.1$'$}$$

$$\sum_{k=0}^{\infty}(k + 1)B(k) < \infty, \quad \text{with} \tag{1.2$'$}$$

$$B(k) = \sup_{\substack{r>0\\ |u|+|v|\le r}} \left(\int_{2^k r\le|x-y|<2^{k+1}r} \Delta K(x + u, x; y + v, y)\, dy\right).$$

The $T(1)$ theorem has also been considered by Lemarié on the Besov spaces [10], Frazier *et al* on the Triebel-Lizorkin spaces [7], and Han and Hofmann on both classes of spaces [8]. The definitions of such spaces can be stated as follows (see [13]):

Let $\mathscr{S}(\mathbb{R}^n)$ be the space of tempered test functions. Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$ with $\operatorname{supp}\hat{\varphi} \subset \{\xi \in \mathbb{R}^n : 1/2 \le |\xi| \le 2\}$ and $|\hat{\varphi}(\xi)| \ge c > 0$ for $3/5 \le |\xi| \le 5/3$; put $\varphi_j(x) = 2^{jn}\varphi(2^j x)$ and $Q_j(f)(x) = \varphi_j * f(x)$. For $\alpha \in \mathbb{R}$ and $0 < p, q < \infty$, the Besov spaces $\dot{B}_p^{\alpha,q}$ is the collection of all $f \in \mathscr{S}'/\mathscr{P}$ (the tempered distributions modulo polynomials) satisfying

$$\|f\|_{\dot{B}_p^{\alpha,q}} = \left(\sum_j \big(2^{j\alpha}\|Q_j f\|_p\big)^q\right)^{1/q} < \infty.$$

The Triebel-Lizorkin space is defined analogously, $\dot{F}_p^{\alpha,q}$ being the collection of all $f \in \mathscr{S}'/\mathscr{P}$ such that

$$\|f\|_{\dot{F}_p^{\alpha,q}} = \left\|\left(\sum_j \big(2^{j\alpha}|Q_j f|\big)^q\right)^{1/q}\right\|_p < \infty.$$

In this paper, we will prove the following two theorems:

THEOREM 1.1. *Suppose T satisfies the WBP (1.3), the kernel $K(x, y)$ satisfies*

(i) (1.1′) *and* (1.2′). *If $T(1) = T^*(1) = 0$, then T is bounded on $\dot{B}_p^{0,q}$, $1 \leq p, q < \infty$.*

(ii) (1.1′) *and* $\sum_{k=0}^{\infty} 2^{k\alpha} B(k) < \infty$. *If $T(1) = 0$, then T is bounded on $\dot{B}_p^{\alpha,q}$, $0 < \alpha < 1$ and $1 \leq p, q < \infty$.*

THEOREM 1.2. *Suppose T satisfies the WBP (1.3), the kernel $K(x, y)$ satisfies*

(i) (1.1′) *and* $\sum_{k=0}^{\infty}(k+1)^{2-1/q} B(k) < \infty$. *If $T(1) = T^*(1) = 0$, then T is bounded on $\dot{F}_p^{0,q}$, $1 \leq p, q < \infty$.*

(ii) (1.1′) *and* $\sum_{k=0}^{\infty} 2^{k\alpha} B(k) < \infty$. *If $T(1) = 0$, then T is bounded on $\dot{F}_p^{\alpha,q}$, $0 < \alpha < 1$ and $1 \leq p, q < \infty$.*

We remark that the two theorems extend the results of Han and Hofmann [8]; they need to assume that $B(k) \leq C2^{-k\epsilon}$ for $0 < \alpha < \epsilon$ in both theorems. For $-1 < \alpha < 0$, Theorem 1.1 and Theorem 1.2 also hold by interchanging the role of $T(1)$ and $T^*(1)$ because of the duality (the dual of $\dot{B}_p^{\alpha,q}$ is $\dot{B}_{p'}^{-\alpha,q'}$ and similarly for $\dot{F}_p^{\alpha,q}$).

Note that $\dot{F}_p^{0,2}$ is of special interest because it is the Hardy space H^1 when $p = 1$ and is L^p when $p > 1$. For the Hardy space H^1, the kernel condition in Theorem 1.2 is $\sum_k (k+1)^{3/2} B(k) < \infty$. In [5], T is proved to be bounded on L^2 under the kernel condition $\sum_k (k+1)^{1/2} B(k) < \infty$. By the interpolation theorem, a direct application of the theorem yields the following result, which is stronger than the corresponding case stated in (i).

COROLLARY 1.3. *Suppose T satisfies the WBP (1.3), the kernel $K(x, y)$ satisfies* (1.1′) *and* $\sum_k (k+1)^{1/2+2|1/p-1/2|} B(k) < \infty$. *If $T(1) = T^*(1) = 0$, then T is bounded on L^p, $1 < p < \infty$.*

The main tool used in proving the theorems is wavelets, initiated in [2] and [5]. This is quite different from the approaches in [7, 8, 10, 11]. The proof of Theorem 1.1 depends on the Beylkin-Coifman-Rohklin wavelet decomposition of the operator T. For Theorem 1.2, we first prove the boundedness of T on $\dot{F}_1^{\alpha,q}$ using an atomic decomposition on this space. This, together with an interpolation on $\dot{F}_p^{\alpha,p} (= \dot{B}_p^{\alpha,p})$, yields the boundedness of T for the other case.

The paper is organized as follows. In Section 2 we will give some preliminaries on wavelets and the BCR decomposition of T. We also set up the proof in terms of wavelet terminology. The $T(1)$ theorem on the Besov spaces is proved in Section 3 and on the Triebel-Lizorkin spaces in Section 4.

2. Preliminaries

For simplicity, we only consider the one dimensional case. The higher dimensional case is similar.

Let us recall the concept of *multiresolution analysis* in $L^2(\mathbb{R})$ [12]: it is an increasing sequence of closed linear subspaces $\{V_j\}_{j\in\mathbb{Z}} \subseteq L^2(\mathbb{R})$ with the following properties:

(i) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$, $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$;

(ii) For every $j \in \mathbb{Z}$ and $f \in L^2(\mathbb{R})$, $f \in V_j \Longleftrightarrow f(2\,\cdot) \in V_{j+1}$;

(iii) There exists a φ in V_0 such that $\varphi(x-k)$, $k \in \mathbb{Z}$, is an orthonormal basis for V_0.

The above φ is called a *scaling function*. Note that by adjusting a normalization constant, $\sum_k \phi(x-k) = 1$ for all $x \in \mathbb{R}$ [3]. For each $j \in \mathbb{Z}$, we define $\varphi_{jk}(x) = 2^{j/2}\varphi(2^j x - k)$, $k \in \mathbb{Z}$. The sequence $\{\varphi_{jk}\}_{k\in\mathbb{Z}}$ forms an orthonormal basis for V_j. From φ we can construct a wavelet function ψ. Then $\{\psi_{jk}\}_{k\in\mathbb{Z}}$ forms an orthonormal basis for W_j, the orthogonal complement to V_j inside V_{j+1}, that is, $V_{j+1} = V_j \oplus W_j$. It follows that $\{\psi_{jk}\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$. In this paper, we assume that the wavelets are compactly supported, say supp φ, supp $\psi \subseteq [0, M]$ for some integer M. Also we assume that they have the desirable degree of smoothness whenever needed.

We need the following characterizations of the Besov and Triebel-Lizorkin spaces [6, 12].

PROPOSITION 2.1. *Suppose $\psi \in C^\gamma$ is a compactly supported wavelet and $\{\psi_{jk}\}_{j,k\in\mathbb{Z}}$ forms an orthonormal basis of $L^2(\mathbb{R})$. Let f be locally integrable and write $f(x) = \sum_{jk} \alpha(j,k)\psi_{jk}(x)$ formally.*

(i) *For $0 < \alpha < \gamma$, $1 \le p, q < \infty$, $f \in \dot{B}_p^{\alpha,q}$ if and only if*

$$\left(\sum_j \left(\sum_k \left|2^{(-1/p+\alpha+1/2)j}\alpha(j,k)\right|^p\right)^{q/p}\right)^{1/q} < \infty.$$

(ii) *For $0 < \alpha < \gamma$, $1 \le p, q < \infty$, $f \in \dot{F}_p^{\alpha,q}$ if and only if*

$$A(f)(x) = \left(\sum_{j,k} 2^{(\alpha+1/2)jq}|\alpha(j,k)|^q \chi(2^j x - k)\right)^{1/q} \in L^p(\mathbb{R}),$$

where χ denotes the characteristic function of $[0, 1)$. In this case, $\|f\|_{\dot{F}_p^{\alpha,q}} \approx \|A(f)\|_p$.

Let $P_j : L^2(\mathbb{R}) \to V_j$ be the orthonormal projection and $Q_j = P_{j+1} - P_j$. Then $Q_j : L^2(\mathbb{R}) \to W_j$ is the corresponding orthonormal projection. In [2], Beylkin, Coifman and Rokhlin give a decomposition of T in terms of P_j and Q_j:

$$T = \sum_{-\infty}^{\infty} P_j T Q_j + \sum_{-\infty}^{\infty} Q_j T P_j + \sum_{-\infty}^{\infty} Q_j T Q_j. \tag{2.1}$$

The corresponding distribution kernel is

$$\begin{aligned} K(x, y) = &\sum_{jkl} a(j, k, l)\varphi_{jk}(x)\psi_{jl}(y) + \sum_{jkl} b(j, k, l)\psi_{jk}(x)\varphi_{jl}(y) \\ &+ \sum_{jkl} c(j, k, l)\psi_{jk}(x)\psi_{jl}(y), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} a(j, k, l) &= \langle T\psi_{jl}, \varphi_{jk}\rangle = \langle K, \varphi_{jk} \otimes \psi_{jl}\rangle, \\ b(j, k, l) &= \langle T\varphi_{jl}, \psi_{jk}\rangle = \langle K, \psi_{jk} \otimes \varphi_{jl}\rangle, \\ c(j, k, l) &= \langle T\psi_{jl}, \psi_{jk}\rangle = \langle K, \psi_{jk} \otimes \psi_{jl}\rangle. \end{aligned}$$

We call such $a(j, k, l)$, $b(j, k, l)$, $c(j, k, l)$ the BCR-coefficients.

It is easy to show that

PROPOSITION 2.2. *Suppose T satisfies the conditions in Theorem 1.1 or Theorem 1.2, then $T^*(1) = 0$ implies that for any $j, l \in \mathbb{Z}$, $\sum_k a(j, k, l) = 0$; similarly $T(1) = 0$ implies that for any $j, k \in \mathbb{Z}$, $\sum_l b(j, k, l) = 0$.*

PROOF. Assuming that $T^*(1) = 0$ and using $\sum_k \varphi(x - k) = 1$, we have

$$\sum_k a(j, k, l) = \sum_k \langle T\psi_{jl}, \varphi_{jk}\rangle = \langle T\psi_{jl}, 2^{j/2}\rangle = \langle \psi_{jl}, 2^{j/2} T^*(1)\rangle = 0.$$

The second part can be proved similarly. □

PROPOSITION 2.3. *Suppose T satisfies the conditions in Theorem 1.1 or Theorem 1.2. Let*

$$A(m) = \sup_{j,l} \left\{ \sum_{k:2^m \le |k-l| < 2^{m+1}} \left(|a(j, k, l)| + |a(j, l, k)| + |b(j, k, l)| + |b(j, l, k)|\right) \right\}.$$

Then there exists C such that $A(m) \le CB(m)$ for all $m \ge 0$.

Moreover we have

$$\sup_{j,l} \sum_k \left(|c(j, k, l)| + |c(j, l, k)|\right) < \infty. \tag{2.3}$$

PROOF. We first observe that $A(m) < \infty$ for each $m \geq 0$. This comes directly from the WBP in (1.3) and the expressions for $a(j, k, l)$ and $b(j, k, l)$. By using this, it suffices to prove the inequality for $2^m > M$.

Let $y_0 = 2^{-j} l$, then

$$\begin{aligned} a(j, k, l) &= \iint K(x, y)\varphi_{jk}(x)\psi_{jl}(y)\, dx\, dy \\ &= 2^j \iint \big(K(x, y + y_0) - K(x, y_0)\big)\varphi(2^j x - k)\psi(2^j y)\, dx\, dy \\ &\leq C \sup_{y\in[0, 2^{-j}M]} \int_{[k2^{-j}, (k+M)2^{-j}]} |K(x, y + y_0) - K(x, y_0)|\, dx. \end{aligned}$$

Hence for $2^m > M$,

$$\sum_{k:2^m \leq |k-l| < 2^{m+1}} |a(j, k, l)| \leq C \sup_{y\in[0, 2^{-j}M]} \int_E |K(x, y + y_0) - K(x, y_0)|\, dx,$$

where $E = \{x \in \mathbb{R} : 2^{m-j} \leq |x - y_0| \leq 2^{m+1-j} + 2^{-j}M\}$. According to the definition of $B(m)$, we have $\sum_{k:2^m \leq |k-l| < 2^{m+1}} |a(j, k, l)| \leq CB(m)$ for $2^m > M$. The same argument applies to the other terms in $A(m)$, which completes the proof for the first assertion.

The proof of (2.3) is essentially the same. We consider

$$\sup_{j,l} \left\{ \sum_{k\in J} \big(|c(j, k, l)| + |c(j, l, k)|\big) \right\}.$$

It is bounded if $J = \{k : |k - l| \leq M\}$; and it is $\leq CB(m)$ if $J = \{k : 2^m \leq |k - l| < 2^{m+1}\}$ if $2^{m+1} > M$. This implies (2.3). $\square$

3. $T(1)$ Theorem on Besov spaces

In view of Proposition 2.2 and Proposition 2.3, we will prove the following theorem in terms of the BCR-coefficients, which implies Theorem 1.1.

THEOREM 3.1. *Let $T : \mathscr{D} \longrightarrow \mathscr{D}'$ be a Calderón-Zygmund operator with the wavelet decomposition as in* (2.1), (2.2) *and satisfying* (1.1′), (2.3).

(i) *If $\sum_{m=0}^{\infty} (m + 1)A(m) < \infty$ and $\sum_k a(j, k, l) = \sum_l b(j, k, l) = 0$, then T is a bounded operator on $\dot{B}_p^{0,q}$, $1 \leq p, q < \infty$.*

(ii) *If $\sum_{m=0}^{\infty} 2^{\alpha m} A(m) < \infty$ and $\sum_l b(j, k, l) = 0$ for any $j, k \in \mathbb{Z}$, then T is bounded on $\dot{B}_p^{\alpha,q}$, $0 < \alpha < 1$, $1 \leq p, q < \infty$.*

Let us rewrite

$$T = \sum_{-\infty}^{\infty} P_j T Q_j + \sum_{-\infty}^{\infty} Q_j T P_j + \sum_{-\infty}^{\infty} Q_j T Q_j = T^{(1)} + T^{(2)} + T^{(3)}.$$

We will first consider the term $T^{(2)}$. Its distributional kernel is

$$K^{(2)}(x, y) = \sum_{jkl} b(j, k, l) \psi_{jk}(x) \varphi_{jl}(y).$$

Let $J_m = \{(k, l) : 2^m \le |k - l| < 2^{m+1}\}$, and

$$b_m(j, k, l) = \begin{cases} b(j, k, l) & (k, l) \in J_m; \\ -\sum_{n:(k,n)\in J_m} b(j, k, n) & l = k; \\ 0 & \text{otherwise}, \end{cases} \tag{3.1}$$

where $m = 0, 1, 2, \ldots$. The definition implies that $\sum_l b_m(j, k, l) = 0$ for each j, $k \in \mathbb{Z}$. Since $\sum_l b(j, k, l) = 0$ by assumption, we have

$$b(j, k, k) = -\sum_{m=0}^{\infty} \sum_{l:(k,l)\in J_m} b(j, k, l).$$

Hence $K^{(2)}(x, y)$ can be decomposed as

$$\begin{aligned} K^{(2)}(x, y) &= \sum_{jk} b(j, k, k) \psi_{jk}(x) \varphi_{jk}(y) \\ &\quad + \sum_{jk} \sum_{l:l\neq k} b(j, k, l) \psi_{jk}(x) \varphi_{jl}(y) \\ &= \sum_{jk} \left(-\sum_{m=0}^{\infty} \sum_{l:(k,l)\in J_m} b(j, k, l) \right) \psi_{jk}(x) \varphi_{jk}(y) \\ &\quad + \sum_{m=0}^{\infty} \sum_{jk} \sum_{l:(k,l)\in J_m} b(j, k, l) \psi_{jk}(x) \varphi_{jl}(y) \\ &= \sum_{m=0}^{\infty} \sum_{jkl} b_m(j, k, l) \psi_{jk}(x) \varphi_{jl}(y) = \sum_{m=0}^{\infty} K_m^{(2)}(x, y). \end{aligned} \tag{3.2}$$

Let $T_m^{(2)}$ denote the operator with distributional kernel $K_m^{(2)}(x, y)$. Then we can decompose $T^{(2)}$ as: $T^{(2)} = \sum_{m=0}^{\infty} T_m^{(2)}$. We call each $T_m^{(2)}$ a *block operator*. The following lemma together with the assumption on $A(m)$ will imply that $T^{(2)}$ is a bounded operator on $\dot{B}_p^{\alpha,q}$.

LEMMA 3.2. *Under the hypothesis of Theorem* 3.1, *let* $1 \le p, q < \infty$, $0 \le \alpha < 1$. *Then* $T_m^{(2)}$ *is a bounded operator on* $\dot{B}_p^{\alpha,q}$, *and the operator norm satisfies*

$$\| T_m^{(2)} \| \le \begin{cases} C(m+1)A(m) & \alpha = 0; \\ C2^{\alpha m}A(m) & 0 < \alpha < 1, \end{cases}$$

where C *is independent of* m.

PROOF. Let $f(y) = \sum_{jk} \alpha(j,k)\psi_{jk}(y)$ be in $\dot{B}_p^{\alpha,q}$, and let $g(x) = \sum_{jk} \beta(j,k)\psi_{jk}(x)$ be in the dual space $\dot{B}_{p'}^{-\alpha,q'}$. Noting that $\langle \varphi_{jl}, \psi_{j'k'} \rangle \neq 0$ implies that $j > j'$ and $2^{j-j'}k' \le l \le 2^{j-j'}(k'+M) + M$ (recall that φ, ψ have compact supports contained in $[0, M]$), one can write

$$\begin{aligned}
\langle T_m^{(2)} f, g \rangle &= \sum_{j'k'} \sum_{jkl} \alpha(j',k') b_m(j,k,l) \beta(j,k) \langle \varphi_{jl}, \psi_{j'k'} \rangle \\
&= \sum_{j,j':0<j-j'\le m} \left(\sum_{kk'l} \alpha(j',k') b_m(j,k,l) \beta(j,k) \langle \varphi_{jl}, \psi_{j'k'} \rangle \right) \\
&\quad + \sum_{j,j':j-j'\ge m+1} \left(\sum_{kk'l} \alpha(j',k') b_m(j,k,l) \beta(j,k) \langle \varphi_{jl}, \psi_{j'k'} \rangle \right) \\
&= \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Let $\Gamma_{kk'l}^{mj's} = b_m(j'+s,k,l)\langle \varphi_{j'+sl}, \psi_{j'k'} \rangle$. By using Proposition 2.1 and the Hölder inequality, we obtain

$$\begin{aligned}
|\mathrm{I}| &\le \sum_{s=1}^{m} \sum_{j'} \left| \sum_{kk'l} \alpha(j',k') b_m(j'+s,k,l) \beta(j'+s,k) \langle \varphi_{j'+sl}, \psi_{j'k'} \rangle \right| \\
&\le \sum_{s=1}^{m} \left\{ \sum_{j'} \left(\sum_{k'} |\alpha(j',k')|^p \right)^{1/p} \left(\sum_{k'} \left| \sum_{kl} \beta(j'+s,k)\, \Gamma_{kk'l}^{mj's} \right|^{p'} \right)^{1/p'} \right\} \\
&\le C \sum_{s=1}^{m} 2^{(1/p'+\alpha-1/2)s} \|f\|_{\dot{B}_p^{\alpha,q}} \\
&\quad \times \left\{ \sum_{j'} \left(\sum_{k'} \left| 2^{(-1/p'-\alpha+1/2)(j'+s)} \sum_{kl} \beta(j'+s,k) \Gamma_{kk'l}^{mj's} \right|^{p'} \right)^{q'/p'} \right\}^{1/q'}.
\end{aligned}$$

We will make a separate estimation of $\sum_{k'} \cdots$. Note that

$$\sum_{kl} \left|\Gamma^{mj's}_{kk'l}\right| \le 2^{-s/2} \sum_{l} \sum_{k} \left|b_m(j'+s,k,l)\left\langle \varphi(x-l), \psi(2^{-s}x-k')\right\rangle\right|$$
$$\le 2^{-s/2} A(m) \sum_{l} \left|\left\langle \varphi(x-l), \psi(2^{-s}x-k')\right\rangle\right|$$
$$\le C 2^{s/2} A(m),$$

(the last inequality holds because φ has compact support and $\sum_l |\varphi(x-l)|$ is bounded) and

$$\sum_{k'l} \left|\Gamma^{mj's}_{kk'l}\right| \le 2^{-s/2} \sum_{l} \left|b_m(j'+s,k,l)\right| \left\langle \left|\varphi(x-l)\right|, \sum_{k'} \left|\psi(2^{-s}x-k')\right| \right\rangle$$
$$\le C 2^{-s/2} A(m).$$

Hence

$$\sum_{k'} \left|\sum_{kl} \beta(j'+s,k)\Gamma^{mj's}_{kk'l}\right|^{p'} \tag{3.3}$$
$$\le \sum_{k'} \left\{ \left(\sum_{kl} \left|\beta(j'+s,k)\right|^{p'} \left|\Gamma^{mj's}_{kk'l}\right| \right) \left(\sum_{kl} \left|\Gamma^{mj's}_{kk'l}\right| \right)^{p'/p} \right\}$$
$$\le C \left(\sum_{k} \left|\beta(j'+s,k)\right|^{p'} \sum_{k'l} \left|\Gamma^{mj's}_{kk'l}\right| \right) \left(2^{s/2}A(m)\right)^{p'/p}$$
$$\le C \left(\sum_{k} \left|\beta(j'+s,k)\right|^{p'} \right) \left(2^{-s/2}A(m)\right) \left(2^{s/2}A(m)\right)^{p'/p}$$
$$= C 2^{s(p'/p-1)/2} |A(m)|^{p'} \sum_{k} \left|\beta(j'+s,k)\right|^{p'}.$$

It follows that

$$|\mathrm{I}| \le C \sum_{s=1}^{m} 2^{(1/p'+\alpha-1/2)s} 2^{s(1/p-1/p')/2} \cdot A(m) \|f\|_{\dot{B}^{\alpha,q}_p}$$
$$\times \left\{ \sum_{j'} \left(\sum_{k} \left| 2^{(-1/p'-\alpha+1/2)(j'+s)} \beta(j'+s,k) \right|^{p'} \right)^{q'/p'} \right\}^{1/q'}$$
$$\le C \sum_{s=1}^{m} 2^{\alpha m} A(m) \|f\|_{\dot{B}^{\alpha,q}_p} \|g\|_{\dot{B}^{-\alpha,q'}_{p'}}$$
$$\le \begin{cases} C(m+1)A(m)\|f\|_{\dot{B}^{\alpha,q}_p}\|g\|_{\dot{B}^{-\alpha,q'}_{p'}} & \alpha = 0; \\ C2^{\alpha m}A(m)\|f\|_{\dot{B}^{\alpha,q}_p}\|g\|_{\dot{B}^{-\alpha,q'}_{p'}} & 0 < \alpha < 1. \end{cases}$$

We now estimate the expression II. For convenience, we use the same notation C to denote the different constants in the different place. Define

$$g_{jk}(x) = \int_{-\infty}^{x} \sum_{l} b_m(j, k, l)\varphi(y-l)dy \quad \text{and} \quad \tilde{\Gamma}_{kk'}^{j's} = \left\langle g_{j'+sk}(x), \psi'(2^{-s}x - k)\right\rangle.$$

Then

$$\begin{aligned}
|\mathrm{II}| &= \left| \sum_{j,j':j-j'\geq m+1} \sum_{kk'} \alpha(j', k')b_m(j, k, l)\beta(j, k)\left\langle \varphi_{jl}, \psi_{j'k'}\right\rangle \right| \\
&= C\sum_{s=m+1}^{\infty} \sum_{j'=-\infty}^{\infty} \sum_{kk'} 2^{-3s/2}\left|\alpha(j', k')\beta(j'+s, k)\tilde{\Gamma}_{kk'}^{j's}\right| \\
&\leq C\sum_{s=m+1}^{\infty} 2^{-3s/2}\|f\|_{\dot{B}_p^{\alpha,q}} \\
&\quad \times \left\{ \sum_{j'} 2^{(-1/p'-\alpha+1/2)j'q'} \left(\sum_{k'} \left| \sum_{k} \beta(j'+s, k)\tilde{\Gamma}_{kk'}^{j's}\right|^{p'}\right)^{q'/p'}\right\}^{1/q'}.
\end{aligned}$$

We claim that

$$\sum_{k'} \left| \sum_{k} \beta(j'+s, k)\tilde{\Gamma}_{kk'}^{j's}\right|^{p'} \leq C2^{(s+m)(p'-1)}2^m|A(m)|^{p'} \sum_{k} \left|\beta(j'+s, k)\right|^{p'}. \tag{3.4}$$

In fact, the condition $\sum_l b_m(j, k, l) = 0$ implies that supp $g_{jk} \subseteq [k-2^{m+1}, k+2^{m+1}M]$. Then

$$\begin{aligned}
\sum_{k'} \left|\tilde{\Gamma}_{kk'}^{j's}\right| &\leq C\left\langle \left|g_{j'+sk}(x)\right|, \sum_{k'} \left|\psi'(2^{-s}x - k')\right| \right\rangle \\
&\leq C\left\langle \left|g_{j'+sk}(x)\right|, 1\right\rangle \leq C2^m A(m).
\end{aligned}$$

On the other hand, for $s = j - j'$ since supp $\psi'(2^{-s}x - k') \subseteq [2^s k', 2^s(k'+M)]$, we know that $|k - 2^s k'| \leq 2^s M + 2^m + M \leq C2^s$. It follows that

$$\sum_{k} \left|\tilde{\Gamma}_{kk'}^{j's}\right| \leq C\sum_{k} \left\langle \left|g_{j'+sk}(x)\right|, \left|\psi'(2^{-s}x - k')\right|\right\rangle \leq C2^{s+m}A(m).$$

Combining these two estimates we have

$$\sum_{k'}\left|\sum_{k}\beta(j'+s,k)\tilde{\Gamma}^{j's}_{kk'}\right|^{p'}$$
$$\leq \sum_{k'}\left(\sum_{k}|\beta(j'+s,k)|^{p'}\left|\tilde{\Gamma}^{j's}_{kk'}\right|\right)\left(\sum_{k}\left|\tilde{\Gamma}^{j's}_{kk'}\right|\right)^{p'/p}$$
$$\leq C2^{(s+m)(p'-1)}2^{m}|A(m)|^{p'}\sum_{k}\left|\beta(j'+s,k)\right|^{p'}.$$

This proves the claim. We return to the estimate of $|\mathrm{II}|$

$$|\mathrm{II}|\leq C\sum_{s=m+1}^{\infty}2^{(1/p'+\alpha-2)s}\|f\|_{\dot{B}^{\alpha,q}_{p}}\left\{\sum_{j'}\left(2^{(s+m)(p'-1)}2^{m}|A(m)|^{p'}\right.\right.$$
$$\left.\left.\times\sum_{k}\left|2^{(-1/p'-\alpha+1/2)(j'+s)}\beta(j'+s,k)\right|^{p'}\right)^{q'/p'}\right\}^{1/q'}$$
$$\leq C\left(\sum_{s=m+1}^{\infty}2^{(\alpha-1)s}2^{m}A(m)\right)\|f\|_{\dot{B}^{\alpha,q}_{p}}\|g\|_{\dot{B}^{-\alpha,q'}_{p'}}$$
$$\leq C2^{\alpha m}A(m)\|f\|_{\dot{B}^{\alpha,q}_{p}}\|g\|_{\dot{B}^{-\alpha,q'}_{p'}}.$$

By the estimates of $|\mathrm{I}|$ and $|\mathrm{II}|$ we conclude that $T^{(2)}$ is a bounded operator on $\dot{B}^{\alpha,q}_{p}$, $1\leq p,q<\infty$, $0\leq\alpha<1$ with the operator norm as specified. □

LEMMA 3.3. *Under the hypothesis of Theorem* 3.1, *let* $1\leq p,q<\infty$, $0\leq\alpha<1$. *Then* $T^{(1)}$ *is a bounded operator on* $\dot{B}^{\alpha,q}_{p}$.

PROOF. We can write

$$\left\langle T^{(1)}f,g\right\rangle=\sum_{jj'kk'}a(j,k,l)\beta(j',k')\alpha(j,l)\left\langle\varphi_{jk},\psi_{j'k'}\right\rangle$$

with $j>j'$ and $2^{j-j'}k'\leq l\leq 2^{j-j'}(k'+M)+M$ as in Lemma 3.2. For the case $\alpha=0$, we have by assumption that $\sum_{k}a(j,k,l)=0$, and we can apply the same proof as in Lemma 3.2 (by replacing $\sum_{l}b(j,k,l)=0$).

For the case $0<\alpha<1$, we have not assumed that $\sum_{k}a(j,k,l)=0$ and we need to modify the proof by separating out the diagonal term (as in (3.2)):

$$
\begin{aligned}
|\langle T^{(1)}f, g\rangle| \le & \left| \sum_{j,j':j-j'>0} \sum_{kk'} \beta(j', k') a(j, k, k) \alpha(j, k) \langle \varphi_{jk}, \psi_{j'k'} \rangle \right| \\
& + \left| \sum_{m=0}^{\infty} \sum_{j,j':j-j'>0} \sum_{(k,l)\in J_m} \sum_{k'} \beta(j', k') a(j, k, l) \alpha(j, l) \langle \varphi_{jk}, \psi_{j'k'} \rangle \right| .
\end{aligned}
$$

By using the same argument as in Lemma 3.2, we can show that the first term is bounded by $C\sum_{s=1}^{\infty} 2^{-\alpha s} \|f\|_{\dot B_p^{\alpha,q}} \|g\|_{\dot B_{p'}^{-\alpha,q'}}$ and the second term is bounded by $CA(m)\sum_{s=1}^{\infty} 2^{-\alpha s} \|f\|_{\dot B_p^{\alpha,q}} \|g\|_{\dot B_{p'}^{-\alpha,q'}}$ (note that in here the term $\sum_{s=1}^{\infty} 2^{-\alpha s}$ converges and we only need to use the estimation for |I| without recoursing to the estimation |II| in the last proof). This proves the lemma. □

LEMMA 3.4. *Under the hypothesis of Theorem* 3.1 *let* $1 \le p, q < \infty$, $0 \le \alpha < 1$. *Then* $T^{(3)}$ *is a bounded operator on the Besov spaces* $\dot B_p^{\alpha,q}$.

PROOF. We can write

$$
\langle T^{(3)}f, g\rangle = \sum_{jkl} c(j, k, l) \alpha(j, k) \beta(j, l),
$$

where f, g are defined as in Lemma 3.2. By using the duality and condition (2.3), it can be checked as in Lemma 3.2 that $|\langle T^{(3)}f, g\rangle| \le C\|f\|_{\dot B_p^{\alpha,q}} \|g\|_{\dot B_{p'}^{-\alpha,q'}}$. □

Theorem 3.1 follows directly from Lemmas 3.2–3.4. □

4. $T(1)$ Theorem on Triebel-Lizorkin spaces

In view of Proposition 2.2 and Proposition 2.3, we will prove the following theorem in terms of the BCR-coefficients, which implies Theorem 1.2.

THEOREM 4.1. *Let* $T : \mathscr{D} \longrightarrow \mathscr{D}'$ *be a Calderón-Zygmund operator with the wavelet decomposition as in* (2.1), (2.2) *and satisfying* (1.1′), (2.3).

(i) *If* $\sum_{m=0}^{\infty} (m+1)^{2-1/q} A(m) < \infty$, *and* $\sum_k a(j, k, l) = \sum_l b(j, k, l) = 0$, *then* T *is a bounded operator on* $\dot F_p^{0,q}$, $1 \le p, q < \infty$.

(ii) *If* $\sum_{m=0}^{\infty} 2^{\alpha m} A(m) < \infty$ *and* $\sum_l b(j, k, l) = 0$ *for any* $j, k \in \mathbb{Z}$, *then* T *is bounded on* $\dot F_p^{\alpha,q}$, $0 < \alpha < 1$, $1 \le p, q < \infty$.

We will first prove the theorem for $\dot F_1^{\alpha,q}$, then apply an interpolation theorem on $\dot{F}_1^{\alpha,q}$ and $\dot{F}_q^{\alpha,q}(=\dot{B}_q^{\alpha,q})$, and a duality argument to conclude the theorem. We need the following notion of atom which can be found in [9].

DEFINITION. Let $a(x)=\sum_{jk}\alpha(j,k)\psi_{jk}(x)$ be a locally integrable function. We say that $a(x)$ is an $(\alpha,1,q)$-*atom* if there is a dyadic cube $I\subset\mathbb{R}$ such that

(i) $\operatorname{supp} a(x)\subset I$;
(ii) $\int_I a(x)\,dx=0$;
(iii) $\|A(a)\|_q\le |I|^{1/q-1}$,

where $A(a)$ is defined by

$$A(a)(x)=\left(\sum_{j,k}2^{(\alpha+1/2)jq}|\alpha(j,k)|^q\chi(2^jx-k)\right)^{1/q}. \tag{4.1}$$

For $j,k\in\mathbb{Z}$, let I_{jk} denote the interval $[2^{-j}k,2^{-j}(k+1)]$. Let

$$a_{jk}(x)=\sum_{j',k':I_{j'k'}\subset I_{jk}}\alpha(j',k')\psi_{j'k'}(x). \tag{4.2}$$

Note that the sum is actually adding all j',k' with $j\le j'$, $k'\in[2^{j'-j}k,2^{j'-j}(k+1))$. It follows that the support of $a_{jk}(x)$ is contained in $[2^{-j}k,2^{-j}(k+M)]$. By the definition, we know that $a_{jk}(x)$ is an $(\alpha,1,q)$-atom if

$$\left\|a_{jk}\right\|_{(\alpha,1,q)}:=\left\{2^{-j(q-1)}\sum_{j',k':I_{j'k'}\subset I_{jk}}2^{(-1/q+\alpha+1/2)j'q}\left|\alpha(j',k')\right|^q\right\}^{1/q}<\infty.$$

LEMMA 4.2. *Let $a_{jk}(x)$ be the atom as in* (4.2), *and let $T_m^{(2)}$ be defined as in Lemma* 3.2. *Then we have*

$$\left\|T_m^{(2)}a_{jk}\right\|_{\dot{F}_1^{\alpha,q}}\le\begin{cases}C(m+1)^{2-1/q}A(m)\left\|a_{jk}\right\|_{(\alpha,1,q)} & \alpha=0;\\ C2^{\alpha m}A(m)\left\|a_{jk}\right\|_{(\alpha,1,q)} & 0<\alpha<1,\end{cases}$$

where C is independent of m,j,k.

PROOF. Without loss of generality we consider $a_{00}(y)$ and denote it by $a(y)$ for simplicity, that is, $a(y)=\sum_{0\le j}\sum_{0\le k<2^j}\alpha(j,k)\psi_{jk}(y)$. Noting that $\langle\varphi_{jl},\psi_{j'k'}\rangle\ne 0$

implies that $j > j'$ and $2^{j-j'}k' \le l \le 2^{j-j'}(k'+M)+M$. One can write

$$\begin{aligned}
\left(T_m^{(2)}a\right)(x) &= \sum_{jkl}\sum_{j'k'} \alpha(j',k')b_m(j,k,l)\left\langle \varphi_{jl}, \psi_{j'k'}\right\rangle \psi_{jk}(x) \\
&= \sum_{j\le m}\left(\sum_{j'kk'l} \alpha(j',k')b_m(j,k,l)\left\langle \varphi_{jl}, \psi_{j'k'}\right\rangle \psi_{jk}(x)\right) \\
&\quad + \sum_{j>m}\left(\sum_{j':0<j-j'\le m}\ \sum_{kk'l} \alpha(j',k')b_m(j,k,l)\left\langle \varphi_{jl}, \psi_{j'k'}\right\rangle \psi_{jk}(x)\right) \\
&\quad + \sum_{j>m}\left(\sum_{j':j-j'>m}\ \sum_{kk'l} \alpha(j',k')b_m(j,k,l)\left\langle \varphi_{jl}, \psi_{j'k'}\right\rangle \psi_{jk}(x)\right) \\
&= a_1(x)+a_2(x)+a_3(x).
\end{aligned}$$

Since $\dot{B}_1^{\alpha,1} = \dot{F}_1^{\alpha,1} \subset \dot{F}_1^{\alpha,q}$, $1 \le q < \infty$, it follows that there exists a constant C such that $\|f\|_{\dot{F}_1^{\alpha,q}} \le C\|f\|_{\dot{B}_1^{\alpha,1}}$. Using Proposition 2.1 and the Hölder inequality, we obtain

$$\begin{aligned}
\|a_1\|_{\dot{F}_1^{\alpha,q}} &\le C\left\|\sum_{j\le m}\sum_{j'kk'l} \alpha(j',k')b_m(j,k,l)\left\langle \varphi_{jl}, \psi_{j'k'}\right\rangle \psi_{jk}\right\|_{\dot{B}_1^{\alpha,1}} \\
&\le C\sum_{j\le m}\sum_{j'kk'l} 2^{(\alpha-1/2)j}\left|\alpha(j',k')b_m(j,k,l)\left\langle \varphi_{jl}, \psi_{j'k'}\right\rangle\right| \\
&\le CA(m)\sum_{0\le j\le m}\ \sum_{0<j'<j}\ \sum_{0\le k'<2^{j'}} 2^{(j-j')/2}2^{(\alpha-1/2)j}\left|\alpha(j',k')\right| \\
&\le CA(m)\left\{\sum_{0\le j\le m} 2^{\alpha j}\left(\sum_{0<j'<m} 2^{-\alpha j' q'}\right)^{1/q'}\right\} \\
&\quad \times\left(\sum_{j'k'} 2^{(-1/q+\alpha+1/2)j'q}\left|\alpha(j',k')\right|^q\right)^{1/q} \\
&\le \begin{cases} C(m+1)^{2-1/q}A(m)\|a\|_{(\alpha,1,q)} & \alpha=0; \\ C2^{\alpha m}A(m)\|a\|_{(\alpha,1,q)} & 0<\alpha<1. \end{cases}
\end{aligned}$$

To estimate $\|a_2\|_{\dot{F}_1^{\alpha,q}}$, we first observe that the condition $0 \le k' < 2^{j'}$ implies that $0 \le 2^{-j'}l \le M$. By the expression of $a_2(x)$, we know that $k \in E_j = [-2^{m+1}, 2^{m+1}+2^j+2^jM]$, and hence

$$\operatorname{supp} a_2(x) \subseteq \bigcup_{j>m}\bigcup_{k\in E_j}[2^{-j}k, 2^{-j}(k+M)] \subseteq [-1, 2M].$$

Let $\Gamma^{mjs}_{kk'l} = b_m(j,k,l)\langle \varphi_{jl}, \psi_{j-sk'}\rangle$. Then

$$\begin{aligned}
\|a_2\|_{\dot F_1^{\alpha,q}} &\le C\left\| \sum_{j>m}\sum_{j':0<j-j'\le m}\sum_{kk'l}\alpha(j',k')b_m(j,k,l)\left\langle \varphi_{jl}, \psi_{j'k'}\right\rangle \psi_{jk}(x)\right\|_{\dot F_1^{\alpha,q}} \\
&\le C\sum_{s=1}^{m}\left\| \sum_{j=m+1}^{\infty}\sum_{kl}\sum_{0\le k'<2^{j-s}} \alpha(j-s,k')\Gamma^{mjs}_{kk'l}\psi_{jk}\right\|_{\dot F_1^{\alpha,q}} \\
&\le C\sum_{s=1}^{m}\int_{\mathbf R}\left(\sum_{j=m+1}^{\infty}\sum_{k}2^{(\alpha+1/2)jq}\left|\sum_{k'l}\alpha(j-s,k')\Gamma^{mjs}_{kk'l}\right|^q \chi(2^jx-k)\right)^{1/q} dx \\
&\le C\sum_{s=1}^{m}\left\{\sum_{j=m+1}^{\infty}\sum_{k}2^{-j}2^{(\alpha+1/2)jq}\left|\sum_{k'l}\alpha(j-s,k')\Gamma^{mjs}_{kk'l}\right|^q\right\}^{1/q}.
\end{aligned}$$

Similar to the estimate (3.3) in Lemma 3.2, we have

$$\sum_k\left|\sum_{k'l}\alpha(j-s,k')\Gamma^{mjs}_{kk'l}\right|^q \le C2^{s(1-q/q')/2}|A(m)|^q\sum_{k'}\left|\alpha(j-s,k')\right|^q.$$

It follows that

$$\begin{aligned}
\|a_2\|_{\dot F_1^{\alpha,q}} &\le CA(m)\sum_{s=1}^{m}\left\{\sum_{j=m+1}^{\infty}2^{-j+(\alpha+1/2)jq}2^{s(1-q/q')/2}\sum_{k'}\left|\alpha(j-s,k')\right|^q\right\}^{1/q} \\
&\le CA(m)\sum_{s=1}^{m}2^{\alpha s}\left\{\sum_{j=m+1}^{\infty}\sum_{k'}2^{(-1/q+\alpha+1/2)(j-s)q}\left|\alpha(j-s,k')\right|^q\right\}^{1/q} \\
&\le \begin{cases} C(m+1)A(m)\|a\|_{(\alpha,1,q)} & \alpha=0; \\ C2^{\alpha m}A(m)\|a\|_{(\alpha,1,q)} & 0<\alpha<1. \end{cases}
\end{aligned}$$

We now turn to estimate the term $a_3(x)$. Like $a_2(x)$, $\operatorname{supp} a_3(x) \subseteq [-1, 2M]$. Let $g_{jk}(x)$ be defined as in the estimate of $|\mathrm{II}|$ of Lemma 3.2 and $\tilde\Gamma^{js}_{kk'} = \langle g_{jk}(x), \psi'(2^{-s}x-k')\rangle$. Then

$$\begin{aligned}
\|a_3(x)\|_{\dot F_1^{\alpha,q}} &\le C\left\| \sum_{j>m}\sum_{j':j-j'>m}\sum_{kk'l}\alpha(j',k')b_m(j,k,l)\left\langle \varphi_{jl}, \psi_{j'k'}\right\rangle \psi_{jk}(x)\right\|_{\dot F_1^{\alpha,q}} \\
&\le C\sum_{s=m+1}^{\infty}2^{-3s/2}\left\|\sum_{j=m+1}^{\infty}\sum_{kk'}\alpha(j-s,k')\tilde\Gamma^{js}_{kk'}\psi_{jk}(x)\right\|_{\dot F_1^{\alpha,q}} \\
&\le C\sum_{s=m+1}^{\infty}2^{-3s/2}\left\{\sum_{j=m+1}^{\infty}\sum_{k}2^{-j+(\alpha+1/2)jq}\left|\sum_{k'}\alpha(j-s,k')\tilde\Gamma^{js}_{kk'}\right|^q\right\}^{1/q}.
\end{aligned}$$

By the same estimate as (3.4) in Lemma 3.2, we obtain

$$\sum_k \left|\sum_{k'} \alpha(j-s,k')\tilde{\Gamma}^{js}_{kk'}\right|^q \le C2^{s+m}2^{mq/q'}\,|A(m)|^q \sum_{k'} \left|\alpha(j-s,k')\right|^q .$$

Hence

$$\begin{aligned}
\|a_3\|_{\dot{F}_1^{\alpha,q}} &\le C \sum_{s=m+1}^{\infty} 2^{-3s/2} \left\{ \sum_{j=m+1}^{\infty} 2^{-j+(\alpha+1/2)jq}2^{s+m}2^{mq/q'} \sum_{k'} \left|\alpha(j-s,k')\right|^q \right\}^{1/q} \\
&\le CA(m) \sum_{s=m+1}^{\infty} 2^m 2^{(\alpha-1)s} \left\{ \sum_{jk'} 2^{(-1/q+\alpha+1/2)(j-s)q} \left|\alpha(j-s,k')\right|^q \right\}^{1/q} \\
&\le C2^{\alpha m}A(m)\|a\|_{(\alpha,1,q)}.
\end{aligned}$$

We have hence proved Lemma 4.2 by combining the estimates of $a_1(x), a_2(x)$ and $a_3(x)$. □

Now we state the following atomic decomposition of $\dot{F}_1^{\alpha,q}$, which is given in [1] for $\dot{F}_1^{0,2} (= H^1)$ and in [6] and [9] for the general case. For completeness we modify their proof and sketch it here.

LEMMA 4.3. *Let $f(x) = \sum_{jk} \alpha(j,k)\psi_{jk}(x)$ be in $\dot{F}_1^{\alpha,q}$. Then there exists a sequence $\{h_{sn}(x)\}_{s,n}$ of $(\alpha, 1, q)$-atoms and $\{\lambda_{sn}\} \in \mathbb{R}$ such that*

$$(4.3)\qquad f(x) = \sum_{s,n} \lambda_{sn}h_{sn}(x) \quad \text{and} \quad C_1\|f\|_{\dot{F}_1^{\alpha,q}} \le \sum_{s,n} |\lambda_{sn}| \le C_2\|f\|_{\dot{F}_1^{\alpha,q}}$$

for some fixed $C_1, C_2 > 0$ independent of f.

PROOF. By Proposition 2.1, $f \in \dot{F}_1^{\alpha,q}$ has an equivalent norm given by $\|f\|_{\dot{F}_1^{\alpha,q}} \approx \|A(f)\|_1$.

If 2^s, $s \in \mathbb{Z}$, is a given threshold, we define $\Omega_s = \{x : A(f)(x) > 2^s\}$. This allows us to write $\Omega_s = \bigcup_{n\in\mathbb{N}} Q_{sn}$, where each Q_{sn} is a maximal dyadic interval in Ω_s. The intervals Q_{sn}, being dyadic and maximal, are either identical or disjoint. For each $s \in \mathbb{Z}$, $n \in \mathbb{N}$, consider the family $\mathscr{F}_{sn}$ of all dyadic intervals I_{jk} such that $I_{jk} \subset Q_{sn}$ which are contained in no Q_{s+1p} for any p. By the above construction, we can write

$$\bigcup_s \Omega_s = \bigcup_{s,n} \left(\bigcup_{I_{jk}\in\mathscr{F}_{sn}} I_{jk} \right).$$

For each $s \in \mathbb{Z}$, $n \in \mathbb{N}$, let

$$h_{sn}(x) = |\lambda_{sn}|^{-1} \sum_{I_{jk} \in \mathcal{F}_{sn}} \alpha(j,k)\psi_{jk}(x), \tag{4.4}$$

where $\lambda_{sn} = |Q_{sn}|^{1-1/q}\left\{\sum_{I_{jk}\in\mathcal{F}_{sn}} 2^{(-1/q+\alpha+1/2)jq}|\alpha(j,k)|^q\right\}^{1/q}$. Then, $h_{sn}(x)$ is an $(\alpha, 1, q)$-atom and

$$f(x) = \sum_{j,k} \alpha(j,k)\psi_{jk}(x) = \sum_{s,n} \lambda_{sn} h_{sn}(x) \tag{4.5}$$

gives an atomic decomposition of f. For the details we refer the reader to [1, 6, 9]. □

Proof of Theorem 4.1. By Lemma 4.4 we can write $f(x) \in \dot{F}_1^{\alpha,q}$ as an atomic decomposition $f(x) = \sum_{s,n} \lambda_{sn} h_{sn}(x)$, where each $h_{sn}(x)$ is an $(\alpha, 1, q)$-atom defined in (4.4). For each $h_{sn}(x)$, we can rewrite it as the form in (4.2) by assigning $\alpha(j,k) = 0$ for $I_{jk} \subset Q_{sn}$ but $I_{jk} \notin \mathcal{F}_{sn}$. Using Lemma 4.2 and Lemma 4.3, we have

$$\| T_m^{(2)} f \|_{\dot{F}_1^{\alpha,q}} \le \begin{cases} C(m+1)^{2-1/q} A(m) \sum_{s,n} |\lambda_{sn}| & \alpha = 0; \\ C 2^{\alpha m} A(m) \sum_{s,n} |\lambda_{sn}| & 0 < \alpha < 1, \end{cases}$$

where C is independent of m, s, n. It follows that $T^{(2)}$ is bounded on $\dot{F}_1^{\alpha,q}$. Similarly as in Lemma 3.3, and Lemma 3.4, we can show that both $T^{(1)}$ and $T^{(3)}$ are also bounded operators on $\dot{F}_1^{\alpha,q}$. Hence, we have proved that T is bounded on $\dot{F}_1^{\alpha,q}$, and

$$\| T \|_{(\dot{F}_1^{\alpha,q}, \dot{F}_1^{\alpha,q})} \le \begin{cases} C \sum_{m=0}^{\infty} (m+1)^{2-1/q} A(m) & \alpha = 0; \\ C \sum_{m=0}^{\infty} 2^{\alpha m} A(m) & 0 < \alpha < 1, \end{cases}$$

where $0 \le \alpha < 1$, $1 \le q < \infty$.

Since T is bounded on $\dot{B}_q^{\alpha,q} (= \dot{F}_q^{\alpha,q})$, the interpolation theorem [13] implies that T is bounded on $\dot{F}_p^{\alpha,q}$, $0 \le \alpha < 1$, $1 \le p \le q$. Similarly as in the proof of Theorem 3.1 and Theorem 4.1, we can show, by interchanging the role of $T(1)$ and $T^*(1)$, that T is bounded on both $\dot{F}_q^{\alpha,q}$ and $\dot{F}_1^{\alpha,q}$ with $-1 < \alpha \le 0$, $1 \le q < \infty$. Again, applying the interpolation theorem and the duality, T is bounded on $\dot{F}_p^{\alpha,q}$, $0 \le \alpha < 1$, $p \ge q$. This finishes the proof of Theorem 4.1. □

Proof of Corollary 1.3. With the above notation, we have $\| T_m^{(2)} \|_{(H^1,H^1)} \le C(m+1)^{3/2} A(m)$ and $\| T_m^{(2)} \|_{(L^2,L^2)} \le C(m+1)^{1/2} A(m)$ for some $C > 0$ independent of m. For $1 < p < 2$, by the interpolation theorem we have $\| T_m^{(2)} \|_{(L^p,L^p)} \le C(m+1)^{1/2+2(1/p-1/2)} A(m)$. Using the duality argument, for $2 < p < \infty$ we have $\| T_m^{(2)} \|_{(L^p,L^p)} \le C(m+1)^{1/2+2|1/p-1/2|} A(m)$. It follows from the assumption on $A(m)$ that $T^{(2)}$ is bounded on L^p. Similarly, $T^{(1)}$ and $T^{(3)}$ are bounded operators on L^p; so is T. This completes the proof of Corollary 1.3. □

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