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ON THE ABSOLUTE CONTINUITY OF A CLASS OF INVARIANT MEASURES

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Abstract. Let X be a compact connected subset of $\mathbb{R}^d$, let $S_j, j = 1, ..., N$, be contractive self-conformal maps on a neighborhood of X, and let $\{p_j(x)\}_{j=1}^N$ be a family of positive continuous functions on X. We consider the probability measure μ that satisfies the eigen-equation

$$\lambda\mu = \sum_{j=1}^{N} p_j(\cdot)\mu \circ S_j^{-1},$$

for some $\lambda > 0$. We prove that if the attractor K is an s-set and μ is absolutely continuous with respect to $\mathcal{H}^s|_K$, the Hausdorff s-dimensional measure restricted on the attractor K, then $\mathcal{H}^s|_K$ is absolutely continuous with respect to μ (i.e., they are equivalent). A special case of the result was considered by Mauldin and Simon (1998). In another direction, we also consider the L^p-property of the Radon-Nikodym derivative of μ and give a condition for which $D\mu$ is unbounded.

1. Introduction

Let X be a compact connected subset of $\mathbb{R}^d$ and let $S_j : X \to X, j = 1, ..., N$, be contractive maps. We call $\{S_j\}_{j=1}^N$ an *iterated function system* (IFS) on X. It is well known that there exists a unique non-empty compact subset $K \subset X$ invariant under $\{S_j\}_{j=1}^N$ in the sense that $K = \bigcup_{j=1}^N S_j(K)$. If we associate with probability weights $\{p_i\}_{i=1}^N$ to the IFS, then there is a unique probability measure μ on X with supp$\mu = K$ satisfying

$$\mu(A) = \sum_{j=1}^{N} p_j \mu \circ S_j^{-1}(A) \tag{1.1}$$

for every Borel set $A \subset X$. As is well known the invariant measure is either continuously singular or absolutely continuous with respect to the Lebesgue measure m on $\mathbb{R}^d$. It is easy to see that if $S_i(K) \cap S_j(K) = \emptyset$, $i \neq j$, then μ must be singular. However, it remains to be a challenging question to determine which is the case if the $S_i(K)$'s have nonempty intersection ([LNR], [PSS]). One of the most basic

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examples of such measures is the classical Bernoulli convolution defined by

$$\mu_\rho = \frac{1}{2}(\mu_\rho \circ S_1^{-1} + \mu_\rho \circ S_2^{-1}),$$

where $S_1(x) = \rho x$ and $S_2(x) = \rho x + (1-\rho)$ and $\rho \in (0,1)$. It is known that μ_ρ is purely singular for $\rho \in (0, 1/2)$ and μ_ρ is absolutely continuous with respect to the Lebesgue measure for *m-a.e.* $\rho \in (1/2, 1)$ (see [PSS] and the references therein). In [MS] Mauldin and Simon proved that if μ_ρ is absolutely continuous with respect to m, then m is also absolutely continuous with respect to μ_ρ, i.e., μ_ρ and m are equivalent. In this paper we will show, among the other results, that the equivalence is actually valid in a more general setting.

Let $\{p_j(\cdot)\}_{j=1}^N$ be a family of positive continuous functions on X associated with a contractive IFS $\{S_j\}_{j=1}^N$. We consider the probability measure μ that satisfies the eigen-equation

$$\lambda\mu = \sum_{j=1}^N p_j(\cdot)\mu \circ S_j^{-1} \tag{1.2}$$

for some $\lambda > 0$. (The notation means $\lambda\mu(A) = \sum_{j=1}^N \int_A p_j(x) d\mu \circ S_j^{-1}(x)$ for every Borel set A.) The measure is associated with the Ruelle-Perron-Frobenius operator $T : C(K) \to C(K)$ and its adjoint $T^* : M(K) \to M(K)$

$$Tf(x) = \sum_{j=1}^N p_j(S_j(x)) f(S_j(x)), \qquad T^*\nu = \sum_{j=1}^N p_j(\cdot)\nu \circ S_j^{-1}, \tag{1.3}$$

where $C(K)$ is the space of continuous functions on K and $M(K)$ is the space of bounded regular Borel measures on K. The operator was introduced by Ruelle (in a more restricted form) to model the Gibbs distribution in statistical mechanics, and was adopted to study the discrete time evolution of flows on the Riemanian manifolds [B]. There has been extensive study on the operator in dynamical system in regard to $\lambda h = Th$ and $\lambda\nu = T^*\nu$. The theory has also been used to study the multifractal structure of measures generated by conformal IFS [MU].

Let D be an open set in $\mathbb{R}^d$. We use C^1 to denote the class of continuously differentiable maps on D. A C^1-map $S : D \to \mathbb{R}^d$ is conformal if $S'(x)$ is a similar matrix, i.e., $S'(x)$ is a positive scalar multiple of an orthogonal matrix. In this case $\|S'(x)\|$, the operator norm of $S'(x)$, is the square root of the maximum eigenvalue of the product of $S'(x)$ and its transpose and equals $|\det S'(x)|^{1/d}$. We say that $\{S_j\}_{j=1}^N$ is a *self-conformal* iterated function system on a compact connected set $X \subset \mathbb{R}^d$ if each S_j extends to an injective map $S_j : D \to D$ on an open neighborhood $D \supset X$ and

$$\sup\{\|S_j'(x)\| : x \in D, j = 1, 2, \cdots, N\} < 1.$$

For the IFS $\{S_j\}_{j=1}^N$, let $J = (j_1, \cdots, j_n) \in \{1, \cdots, N\}^n$ and let $S_J = S_{j_1} \circ \cdots \circ S_{j_n}$. The conformal IFS is said to have the *bounded distortion property* (BDP) if there exists a constant $C > 0$ such that for any index J

$$\frac{\|S_J'(x)\|}{\|S_J'(y)\|} \le C \quad \text{for any} \quad x, y \in D.$$

It is easy to see that if $\{S_j\}_{j=1}^N$ are affine maps, then it has the BDP. Moreover, by adopting the proof in [FL, Lemma 2.3], we can show that $\{S_j\}_{j=1}^N$ also has the

BDP if $\log \|S_j'\|, j = 1, \cdots, N$, satisfy the *Dini condition*, i.e.,

$$\int_0^a \frac{\Omega(\log \|S_j'\|, t)}{t} dt < \infty$$

for some $a > 0$, where $\Omega(\psi, t) := \max\{|\psi(x) - \psi(y)| : |x - y| \le t\}$.

Let $\mathcal{H}^s$ and $\mathcal{H}^s|_K$ be the Hausdorff s-dimensional measure on $\mathbb{R}^d$ and its restriction on the set K respectively. Recall that a set $E \subset \mathbb{R}^d$ is called an s-set if $0 < \mathcal{H}^s(E) < \infty$.

Theorem 1.1. *Suppose that $\{S_j\}_{j=1}^N$ is a self-conformal iterated function system defined on X and has the BDP. If the attractor K is an s-set and the measure μ in* (1.2) *is absolutely continuous with respect to $\mathcal{H}^s|_K$, then in reverse, $\mathcal{H}^s|_K$ is also absolutely continuous with respect to μ on K.*

This generalizes the results in [MS], [PSS, Proposition 3.1] and [HL, Proposition 1.2] where μ is a self-similar measure and $\mathcal{H}^s$ is the Lebesgue measure. It is known that if the conformal $\{S_j\}_{j=1}^N$ has Hölder continuous differential, and s is the unique solution of the Bowen equation $P(s) = 0$, where

$$P(t) = \lim_{n\to\infty} \frac{1}{n} \log \sup_{x\in K} \sum_{|J|=n} \|S_J'(x)\|^t \qquad \text{for } t > 0,$$

then K is an s-set if and only if $\{S_j\}_{j=1}^N$ satisfies the open set condition [PRSS]. (If each S_j is a similitude with a contraction ratio ρ_i, then $P(s) = 0$ is equivalent to the well known formula $\sum_{i=1}^N \rho_i^s = 1$.) If $s = d$, then the absolute continuity of μ with respect to $\mathcal{H}^s|_K$ implies that K is an s-set.

For the absolutely continuous μ, we also have the following interesting results on the equivalence of the local and global L^p-property and on a sufficient condition for the unboundedness of the Radon Nikodym derivative $D\mu$. Let $B_\delta(x)$ represent the open ball centered at x with radius δ.

Theorem 1.2. *Suppose that $\{S_j\}_{j=1}^N$ are contractive one-to-one C^1-maps defined on X and the measure μ in* (1.2) *is absolutely continuous with respect to the Lebesgue measure m. If there exists $x \in K$ and $\delta > 0$ such that $D\mu \in L^p(B_\delta(x) \cap K)$ for some $1 \le p \le \infty$, then $D\mu \in L^p(K)$.*

Theorem 1.3. *Let $\{S_j\}_{j=1}^N$ be contractive one-to-one C^1-maps defined on X and let $p_j(x) = p_j, j = 1, \cdots, N$, be probability weights. If the invariant measure μ in* (1.1) *is absolutely continuous with respect to the Lebesgue measure m and there is at least one $p_j > \beta_j$, where $\beta_j = \max_{x\in K}\{|\det S_j'(x)|\}$, then $D\mu$ is unbounded on the attractor K.*

We will prove these theorems in Section 2 and make some remarks in Section 3.

2. Proof of the theorems

Let $K = \bigcup_{j=1}^N S_j(K)$ be the attractor of the IFS and let $K_J = S_J(K)$, $J = (j_1, \cdots, j_n) \in \{1, 2, \cdots, N\}^n$. It is easy to see that for each n, $K = \bigcup_{|J|=n} K_J$. We let $\mathcal{K} = \{K_J : |J| = n, n \in \mathbb{N}\}$. Then $\mathcal{K}$ is a countable family of compact subsets with the following properties:

(P1) For any $\delta > 0$, there are only finitely many members of $\mathcal{K}$ whose diameters are $> \delta$.

(P2) For any $\varepsilon > 0$, $x \in K$, there exists $K_J \in \mathcal{K}$, such that $x \in K_J \subset B_\varepsilon(x) \cap K$.

We first show that for any open set U, $\mathcal{K}$ has a finite or countable disjoint subfamily which covers $U \cap K$ except for an $\mathcal{H}^s$-zero set.

Lemma 2.1. *Suppose that the IFS defined on X is self-conformal and has the BDP. If the attractor K is an s-set, then for any open set $U \subset \mathbb{R}^d$, there exists a finite or countable disjoint subfamily $\mathcal{G}$ of $\mathcal{K}$ contained in U and $\mathcal{H}^s((U \cap K) \setminus G) = 0$, where $G = \bigcup \mathcal{G}$.*

Proof. Let $\mathcal{K}_U = \{A \in \mathcal{K} \colon A \subseteq U\}$. Using the Vitali covering theorem [F, Theorem 1.10], we can select a finite or countable disjoint subfamily $\mathcal{G}$ of $\mathcal{K}_U$ such that either $\sum_{V \in \mathcal{G}} (\mathrm{diam}(V))^s = \infty$ or $\mathcal{H}^s((U \cap K) \setminus G) = 0$. In the following, we will exclude the first case to complete the proof.

Notice that each S_j extends to an injective map on an open bounded set D which is also connected. Let $\delta_0 = \inf\{|x - y| : x \in X, y \notin D\}$. Then the Mean Value Theorem and the property of conformal map imply that for all $J \in \bigcup_{n=1}^{\infty} \{1, 2, \cdots, N\}^n$ and $x, y \in X$ with $|x - y| < \delta_0$,

$$\min_{u \in D} \|S'_J(u)\| \cdot |x - y| \leq |S_J(x) - S_J(y)| \leq \max_{u \in D} \|S'_J(u)\| \cdot |x - y|. \tag{2.1}$$

Using this bi-Lipschitz property we obtain

$$\left(\min_{u \in D} \|S'_J(u)\|\right)^s \mathcal{H}^s(K) \leq \mathcal{H}^s(S_J(K)) \leq \left(\max_{u \in D} \|S'_J(u)\|\right)^s \mathcal{H}^s(K). \tag{2.2}$$

We will show that the second inequality in (2.1) holds even if $|x - y| \geq \delta_0$. In fact, since D is connected and X is bounded, we can find M balls of radius δ_0 contained in D such that their union is connected and covers X. We then select $B_i = B_{\delta_0}(x_i)$, $i = 1, 2, \cdots, m$, $(m \leq M)$ from the covering such that $x \in B_1$, $y \in B_m$, $B_i \cap B_{i+1} \neq \emptyset$, $i = 1, 2, \cdots, m - 1$. Using the Mean Value Theorem we obtain

$$\begin{aligned} |S_J(x) - S_J(y)| &\leq |S_J(x) - S_J(x_1)| + \sum_{i=1}^{m-1} |S_J(x_i) - S_J(x_{i+1})| + |S_J(x_m) - S_J(y)| \\ &\leq 2(m+1)\delta_0 \max_{u \in D} \|S'_J(u)\| \\ &\leq 2(M+1) \max_{u \in D} \|S'_J(u)\| |x - y|. \end{aligned}$$

It follows that

$$\mathrm{diam}(S_J(K)) \leq 2(M+1) \max_{u \in D} \|S'_J(u)\| \mathrm{diam}(K).$$

Applying the BDP, (2.2) and the above inequality, we can find a constant $C > 0$ such that

$$\mathrm{diam}(S_J(K))^s \leq C\mathcal{H}^s(S_J(K)).$$

This implies that

$$\sum_{V \in \mathcal{G}} (\mathrm{diam}(V))^s \leq C \sum_{V \in \mathcal{G}} \mathcal{H}^s(V) \leq C\mathcal{H}^s(U \cap K) < \infty.$$

∎

Proof of Theorem 1.1. Suppose otherwise there exists a Borel subset $E \subset K$ such that $\mu(E) = 0$ but $\mathcal{H}^s|_K(E) > 0$. Using (1.2) we have

$$0 = \lambda\mu(E) = \sum_{j=1}^{N} \int_E p_j(x) d\mu \circ S_j^{-1}(x) = \sum_{j=1}^{N} \int_{S_j^{-1}(E)} p_j \circ S_j(x) d\mu(x).$$

Since the p_j's are positive functions, $\mu(S_j^{-1}(E)) = 0$ for all j. Let

$$Z = \bigcup_{k=0}^{\infty} \bigcup_{|J|=k} S_J^{-1}(E \cap K_J).$$

It follows that $\mu(Z) = 0$. Note that $Z \subset K$, so $\mu(K \setminus Z) = \mu(K)$. Let us denote $\mathcal{H}^s|_K$ by ν for short. We claim that $\nu(K \setminus Z) = 0$. This will imply that μ is concentrated on a ν-zero subset of K, so ν and μ are mutually singular. It contradicts the hypothesis that μ is absolutely continuous with respect to ν on K, and completes the proof of the theorem.

To prove the claim we note that $\nu(E) > 0$, hence we can apply the density theorem to find a point $x \in E$ such that for any $\varepsilon > 0$, there exists an open ball $B_r(x)$ with

$$\frac{\nu(B_r(x) \cap E)}{\nu(B_r(x))} \geq 1 - \varepsilon.$$

Replacing the U in Lemma 2.1 by $B_r(x)$, we can find a finite or countable disjoint subfamily $\mathcal{G}$ of $\mathcal{K}$ such that each member of $\mathcal{G}$ is a subset of $B_r(x) \cap K$ and

$$\nu((B_r(x) \cap K) \setminus G) = 0,$$

where $G = \bigcup \mathcal{G}$. Note that since $G \subset B_r(x)$, we have

$$\frac{\nu(G \cap E)}{\nu(G)} = \frac{\nu(B_r(x) \cap G \cap E)}{\nu(B_r(x) \cap G)} \geq 1 - \varepsilon.$$

Since members of $\mathcal{G}$ are disjoint, there exists $K_J \in \mathcal{G}$ with

$$\frac{\nu(K_J \cap E)}{\nu(K_J)} \geq 1 - \varepsilon.$$

Observe that since $S_J^{-1}(K_J \cap E) \subset Z$, we have $\nu(S_J Z) \geq \nu(K_J \cap E) \geq (1-\varepsilon)\nu(K_J)$. Since $Z \subset K$, it gives

$$\nu(S_J(K \setminus Z)) = \nu(K_J) - \nu(S_J Z) \leq \varepsilon\nu(K_J).$$

Inequality (2.2) implies that

$$\left(\min_{u \in D} \|S_J'(u)\|\right)^s \nu(K \setminus Z) \leq \nu(S_J(K \setminus Z))$$

and

$$\nu(K_J) \leq \max_{u \in D} \left(\|S_J'(u)\|\right)^s \nu(K).$$

By the BDP it follows that $\nu(K \setminus Z) \leq \varepsilon C \nu(K)$. Since ε is arbitrary, $\nu(K \setminus Z) = 0$ and the claim is proved. $\square$

Proof of Theorem 1.2. Let $f = D\mu$. For any $x \in K, \varepsilon > 0$ and $M > 0$, denote

$$A(x, \varepsilon, M) = \{t \in K \cap B_\varepsilon(x) : \ f(t) > M\}.$$

We first consider the case $p = \infty$. It suffices to show the claim: If $f \notin L^\infty(K)$, then for any given $M_0 > 0, \varepsilon_0 > 0$ and $x_0 \in K$, we have $m(A(x_0, \varepsilon_0, M_0)) > 0$.

For this we first differentiate (1.2) with respect to the Lebesgue measure and get

$$\lambda f(x) = \sum_{j=1}^{N} p_j(x)|\det((S_j^{-1})'(x))| f(S_j^{-1}(x)).$$

Since S_j is contractive and one-to-one, hence $|\det((S_j^{-1})'(x))| > 1$. This implies that for every j

$$\lambda f(x) \geq p_j(x) f(S_j^{-1}(x)). \tag{2.3}$$

Given any $M > 0$, by assumption $f \notin L^\infty(K)$, $m\{t \in K : \ f(t) > M\} > 0$. Using the Lebesgue density theorem, there exists $x^* \in K$ such that for any $\varepsilon > 0$,

$$m(A(x^*, \varepsilon, M)) > 0.$$

Let $x \in S_j(A(x^*, \varepsilon, M))$. Then $x = S_j(t)$ for some $t \in K \cap B_\varepsilon(x^*)$ and $f(t) > M$. Note that S_j is contractive, so $x \in K \cap B_\varepsilon(S_j(x^*))$. Let $0 < \alpha_j =: \min_{x \in K} p_j(x)$. By (2.3) we have

$$f(x) \geq \lambda^{-1} p_j(x) f(S_j^{-1}(x)) \geq \lambda^{-1}\alpha_j M.$$

It follows that

$$S_j(A(x^*, \varepsilon, M)) \subset A(S_j(x^*), \varepsilon, \lambda^{-1}\alpha_j M).$$

Hence

$$\begin{aligned} m(A(S_j(x^*), \varepsilon, \lambda^{-1}\alpha_j M)) &\geq m(S_j(A(x^*, \varepsilon, M))) \\ &= \int_{A(x^*, \varepsilon, M)} |\det S_j'(x)| \, dx \\ &\geq \left(\min_{x \in K} |\det S_j'(x)| \right) m(A(x^*, \varepsilon, M)) \\ &> 0. \end{aligned}$$

By repeating this process, we can prove that for any $J = j_1 \cdots j_n \in \{1, \cdots, N\}^n$ and for any $\varepsilon > 0$,

$$m(A(S_J(x^*), \varepsilon, \lambda^{-n}\alpha_J M)) > 0,$$

where $\alpha_J = \alpha_{j_1} \cdots \alpha_{j_n}$.

Now for any fixed $x_0 \in K, \varepsilon_0 > 0$ and $M_0 > 0$, let $J_0 \in \bigcup_{k \geq 1}\{1, \cdots, N\}^k$ be such that $|S_{J_0}(x) - x_0| < \varepsilon_0/2$ for all $x \in K$. We choose $\varepsilon = \varepsilon_0/2$, $M = \lambda^{|J_0|}\alpha_{J_0}^{-1} M_0$. Then from the above, we have $x^* \in K$ such that

$$m(A(S_{J_0}(x^*), \varepsilon_0/2, M_0)) > 0.$$

By the definition of J_0, we have $|S_{J_0}(x^*) - x_0| < \varepsilon_0/2$. It follows that

$$A(S_{J_0}(x^*), \varepsilon_0/2, M_0) \subset A(x_0, \varepsilon_0, M_0).$$

Thus

$$m(A(x_0, \varepsilon_0, M_0)) \geq m(A(S_{J_0}(x^*), \varepsilon_0/2, M_0)) > 0.$$

The claim is proved.

For the case $1 \leq p < \infty$, we let $C > 0$ be such that $p_j(x) \geq C$ for all $x \in K, j = 1, \cdots, N$. By (2.3), $\lambda^p f^p(x) \geq C^p f^p(S_j^{-1}(x))$. Note that since $f(x)$ is supported on K, we have

$$\begin{aligned}\infty &> \lambda^p \int_{B_\delta(x)} f^p(t)dt \geq C^p \int_{B_\delta(x)} f^p(S_j^{-1}(t))dt \\ &= C^p \int_{S_j^{-1}(B_\delta(x))} f^p(u) \mid \det S_j'(u) \mid du \\ &\geq C^p \min_{t\in K} \mid \det S_j'(t) \mid \int_{S_j^{-1}(B_\delta(x))} f^p(u)du.\end{aligned}$$

Since $\min_{t\in K} |\det S_j'(t)| > 0$, we conclude that

$$\int_{S_j^{-1}(B_\delta(x))} f^p(u)du < \infty.$$

By replacing $B_\delta(x)$ with $S_i^{-1}(B_\delta(x))$, we can show inductively that for all $J \in \{1, \cdots, N\}^n$

$$\int_{S_J^{-1}(B_\delta(x))} f^p(t)dt < \infty.$$

The theorem thus follows from the fact that there exists J such that $S_J(K) \subset B_\delta(x)$, and hence $K \subset S_J^{-1}(B_\delta(x))$. □

Proof of Theorem 1.3. Let $f = D\mu$. It suffices to show that $m\{x \in K : f(x) > M\} > 0$ for any fixed $M > 0$.

For any $J = (j_1, \cdots, j_n)$, let $p_J = p_{j_1} \cdots p_{j_n}$. By using (1.1) repeatedly, we have for any $n \in \mathbb{N}$

$$\begin{aligned}\mu(A) &= \sum_{j=1}^{N}\sum_{i=1}^{N} p_i p_j \mu \circ S_i^{-1} \circ S_j^{-1}(A) \\ &= \sum_{|J|=n} p_J \mu \circ S_J^{-1}(A).\end{aligned}$$

It follows that for any fixed $J = (j_1, \cdots, j_n)$,

$$p_J = p_{j_1} \cdots p_{j_n} \leq \mu(K_J). \tag{2.4}$$

Let j be the index such that $p_j > \beta_j$ as in the hypothesis. Then for any $M > 0$, there exists n_0 such that for all $n \geq n_0$,

$$M\ m(K)\beta_j^n < p_j^n.$$

Let $J^* = (j, j, \cdots, j)$ have length $n \geq n_0$; then

$$m(K_{J^*}) = \int_K |\det S_{J^*}'(x)|dx \leq \beta_j^n m(K).$$

On the other hand, we have

$$\begin{aligned}\mu(K_{J^*}) &= \int_{K_{J^*}} f(x)dx \\ &= \int_{K_{J^*}\cap\{f(x)>M\}} f(x)dx + \int_{K_{J^*}\cap\{f(x)\le M\}} f(x)dx \\ &\le \int_{K_{J^*}\cap\{f(x)>M\}} f(x)dx + M\cdot m(K_{J^*}) \\ &< \int_{K_{J^*}\cap\{f(x)>M\}} f(x)dx + p_j^n.\end{aligned}$$

By (2.4) we have

$$\int_{K_{J^*}\cap\{f(x)>M\}} f(x)dx > \mu(K_{J^*}) - p_j^n \ge 0,$$

so that $m\{x \in K_{J^*} : f(x) > M\} > 0$. This proves the theorem. □

3. Some remarks

Let μ be the Bernoulli convolution defined by $\mu = p\mu \circ S_1^{-1} + (1-p)\mu \circ S_2^{-1}$, where $S_1 x = \rho x, S_2 x = \rho x + (1-\rho)$. In [PS] Peres and Solomyak proved that if $p \in [\frac{1}{3}, \frac{2}{3}]$, then μ is absolutely continuous for almost all $\rho \in [p^p(1-p)^{1-p}, 1)$. Note that $(\frac{1}{3})^{1/3}(\frac{2}{3})^{2/3} = \frac{2^{2/3}}{3} \approx 0.5291 < \frac{2}{3}$. If we take $\frac{2^{2/3}}{3} \le \rho < p = \frac{2}{3}$, we see from Theorem 1.3 that there are invariant measures with unbounded density. We also note that there are $\rho \in [\frac{2^{2/3}}{3}, \frac{2}{3}]$ such that ρ^{-1} is a P.V. number, for such ρ the invariant measure will be purely singular for any weight [LNR].

Theorem 1.2 implies that the μ in (1.2) has a self-similar property that if $D\mu$ is unbounded on K, then it is unbounded in $B_\delta(x)$ for every $x \in \text{supp}\mu$ and for every $\delta > 0$. It is clear that the L^p-property in the theorem cannot be replaced by the C^k-property, namely, the fact that $D\mu$ is C^k-differentiable on a ball does not imply that it is C^k-differentiable on K. For example, let $S_j(x) = x/2 + j$, $j = 0, 1, 2$, with weights $p_0 = p_2 = 1/4$ and $p_1 = 1/2$. Then $D\mu$ is a tent function on [0,4], which has continuous derivatives of all order in $(0, 4) \setminus \{2\}$, but it is not differentiable at 2.

In regard to the question of the eigen-measure in (1.2) being of pure type, we see that if μ is the unique measure satisfying (1.2), then it is either discrete or singularly continuous or absolutely continuous. This can easily be proved by writing μ as the three components and show that each component satisfies (1.2) and then applying the uniqueness of the eigen-measure to make the conclusion. Nevertheless the measure of (1.2) may not be unique if $p_i(x)$ is merely assumed to be continuous [Q], and stronger continuity assumption has to be added [FL]. Therefore we do not have a complete answer for the pure type.

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