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Ruelle operator with nonexpansive IFS

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Abstract. The Ruelle operator and the associated Perron–Frobenius property have been extensively studied in dynamical systems. Recently the theory has been adapted to iterated function systems (IFS) $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$, where the w_j's are contractive self-maps on a compact subset $X \subseteq \mathbb{R}^d$ and the p_j's are positive Dini functions on X [FL]. In this paper we consider Ruelle operators defined by weakly contractive IFS and nonexpansive IFS. It is known that in such cases, positive bounded eigenfunctions may not exist in general. Our theorems give various sufficient conditions for the existence of such eigenfunctions together with the Perron–Frobenius property.

1. Introduction. In [R] Ruelle introduced a convergence theorem to study the equilibrium state (Gibbs measure) of the infinite one-dimensional lattice gas. In [B] Bowen set up the theorem as the convergence of the iterations of a certain operator on the space of continuous functions on a symbolic space. More precisely, let $\Sigma = \{1, \ldots, N\}^{\mathbb{N}}$, let θ be the left shift on Σ and let ϕ be a Hölder continuous function on Σ (the potential function). The *Ruelle operator* is defined as

$$\mathcal{T} f(x) = \sum_{y \in \theta^{-1}(x)} e^{\phi(y)} f(y), \qquad f \in C(\Sigma). \tag{1.1}$$

It was proved that $\mathcal{T}$ has a unique positive eigenfunction $h \in C(\Sigma)$ and a unique probability eigenmeasure $\mu \in C^*(\Sigma)$ corresponding to the spectral radius ϱ, and $h\mu$ is the Gibbs measure (see e.g. [B]). Moreover for any $f \in C(\Sigma)$, $\varrho^{-n}\mathcal{T}^n(f)$ converges uniformly to a constant multiple of h. We will call this the *PF-property* (PF stands for Perron–Frobenius). This theorem together with the theory of Markov partitions was used by Bowen [B]

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to study the ergodic properties of Axiom A diffeomorphisms. Nowadays, the theorem is a standard tool in dynamical systems, thermodynamic formalism and multifractal formalism. There is a vast literature on the Ruelle operator and the related eigenproblem and the convergence property. Ferrero and Schmitt [FS] used the Hilbert projective metric to give yet another proof of Ruelle's theorem. Walters [W] used the g-measure [K] to study the operator and showed that the theorem also holds for Dini continuous potentials, Fan [F1] gave a short proof of the theorem. Quas [Q] gave an example that the eigenmeasure is not unique if we just assume positivity and continuity of p_j's. Mauldin and Urbański [MU1] used the Ruelle operator to study the Hausdorff dimension of the invariant set of a contractive self-conformal system. In [FL] Fan and Lau continued to study the operator by adopting the iterated function system (IFS) point of view: Let $\{w_j\}_{j=1}^m$ be an IFS of contractive self-maps on a compact subset $X \subseteq \mathbb{R}^d$, then there exists a unique compact subset K invariant under the IFS (i.e., $K = \bigcup_{j=1}^m w_j(K)$). With each w_j we associate a positive Dini function p_j as a weight function (or potential function), and we define the Ruelle operator on $C(K)$ as

$$T(f)(x) = \sum_{j=1}^{m} p_j(w_j(x))f(w_j(x)), \qquad f \in C(K). \tag{1.2}$$

It is easy to show that such a T is semi-conjugate to the T in (1.1), and it is conjugate if $w_i(K) \cap w_j(K) = \emptyset$ for $i \neq j$. It was proved that the PF-property holds in this new setting and the Gibbs property [B] of the eigenmeasure μ will also hold if the system consists of contractive self-conformal maps and satisfies the open set condition (OSC) [FL].

Recently a lot of attention is focused on parabolic IFS and nonhyperbolic dynamical systems ([Hu], [LSV], [MU2], [U], [Y], [Yu]), in particular on interval maps with indifferent fixed points ([Hu], [LSV], [PS], [SSU]). It is known that the eigenfunction of the spectral radius ϱ of T may not exist [LY] and even if it exists, ϱ may not be an isolated point of the spectrum [BDE]. So far the available results are far from satisfactory and a study of such systems remains a challenge. We will consider the situation when the $\{w_j\}_{j=1}^m$ are weakly contractive (i.e., $\alpha_{w_j}(t) := \sup_{|x-y|\leq t} |w_j(x)-w_j(y)| < t$ for all $t > 0$) or nonexpansive (i.e., $|w_j(x) - w_j(y)| \leq |x - y|$). For the weakly contractive case, the invariant set K exists as in the contractive case [H]. For the nonexpansive case we can take the smallest invariant K (see Proposition 2.1 for the additional assumption). We can define the Ruelle operator on $C(K)$ as in (1.2). Our first result is (Proposition 2.6):

PROPOSITION 1.1. *Let* $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ *be a weakly contractive system. Suppose* $\alpha_{w_j}(t) \leq t(1-t^\alpha)$ *for* $0 < t < 1$, *and* $\alpha_{\log p_j}(t) = O(t^\beta)$ *for some* $0 < \alpha < \beta \leq 1$. *Then* T *has the PF-property.*

Such special weakly contractive systems are the simplest because the method of proof is the same as in [FL]: we can show that the system is semiconjugate to a symbolic system with a Dini potential function, hence the PF-property of T is inherited from $\mathcal{T}$ on the symbolic space. In general if we only assume that the p_j's are Dini continuous, or even Hölder continuous, we cannot lift the system to a symbolic system with a Dini continuous potential. Hence we will not recourse to the symbolic system in our main considerations. Our basic result is (a special case of Theorem 4.4).

THEOREM 1.2. *Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be a nonexpansive Dini system (i.e., the p_j's are Dini continuous) and let*

$$r_j = \sup_{x \neq y} |w_j(x) - w_j(y)|/|x - y|.$$

Suppose $\| \sum_{j=1}^N p_j \circ w_j(\cdot) r_j \| < \varrho$. Then T has the PF-property.

Note that the condition of this theorem is similar to the *average contractivity* condition of Barnsley *et al.* [BDE] who assumed that $\sum_{j=1}^m p_j(x) = 1$, hence $\varrho = 1$. The condition of Theorem 1.2 is also similar to the one given by Hennion [Hen], but he considered the case that each $p_j(\cdot)$ is Lipschitz continuous. Regarding T as defined on the Lipschitz space, he showed that the essential spectral radius $\varrho_{\text{ess}}(T)$ is strictly less than the spectral radius $\varrho(T)$ and thus T has the PF-property. However his method does not work for the Dini case, since $\varrho(T)$ is not an isolated point of the spectrum in general. By using Theorem 1.2 we prove

THEOREM 1.3. *Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be a nonexpansive Dini system and suppose that $w_1, \dots, w_l$ are contractive for some $1 \le l \le m$. Then $\| \sum_{j=l+1}^m p_j \circ w_j \| < \varrho$ implies that T has the PF-property.*

THEOREM 1.4. *Suppose $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ is a weakly contractive self-conformal Dini system which satisfies the OSC. If $w_1, \dots, w_l$ are contractive for some $0 \le l \le m$ and $\max_{l+1 \le j \le m} \|p_j \circ w_j\| < \varrho$, then T has the PF-property.*

The main idea of the proof of the theorems is laid down in Proposition 3.1 and Lemma 3.3 on the boundedness and equicontinuity of $\{\varrho^{-n} T^n f\}_{n=1}^\infty$.

We remark that the last theorem was considered by Öberg [O] for $X = [0, 1]$ and the p_j's Hölder continuous. In general it is difficult to check the spectral radius condition in the above theorems. Strichartz *et al.* [STZ] have considered a numerical algorithm to approximate the spectral radius ϱ. On the other hand, we see that $\min_{x \in K} \sum_{j=1}^m p_j(w_j x)$ is a lower bound of ϱ; hence if we replace ϱ by $\min_{x \in K} \sum_{j=1}^m p_j(w_j x)$ in the above theorems, we get some simple checkable sufficient conditions.

By using the example of Lasota and Yorke [LY], it is seen that the Ruelle operator may not have an eigenfunction corresponding to the spectral radius (Section 4). However if we enlarge the space $C(K)$ to admit unbounded continuous functions, then an unbounded eigenfunction may exist. E.g., suppose $X = [0,1]$ and a weakly contractive IFS has $w_1(0) = 0$. Let $E = (0,1] \cap K$ and let $C(E)$ be the set of continuous functions on E (including the unbounded ones). In this setup, we can still define the Ruelle operator. Indeed this has been studied in [Hu], [LSV], [Y] as non-hyperbolic dynamical systems. We will consider the unbounded case in a forthcoming paper [LYe].

The present paper is organized as follows. In Section 2, we present some elementary facts about the Ruelle operator and prove Proposition 1.1. We introduce the PF-property in Section 3 and set up basic criteria for this property. We prove Theorems 1.2 and 1.3 in Section 4 and Theorem 1.4 in Section 5.

2. Preliminaries. We consider iterated function systems (IFS) $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ where $X \subseteq \mathbb{R}^d$ is a compact subset, $w_j : X \to X$ are continuous maps and the p_j are positive weight functions (or potential functions) associated with w_j. We say that $w : X \to X$ is *nonexpansive* if $|w(x) - w(y)| \le |x - y|$, and *weakly contractive* if

$$\alpha_w(t) := \sup_{|x-y|\le t} |w(x) - w(y)| < t \qquad \forall t > 0.$$

It is clear that contractivity implies weak contractivity, which also implies nonexpansiveness. A simple nontrivial example of a weakly contractive map is $w(x) = x/(1+x)$ on $[0,1]$. We will call $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ a *weakly contractive system* if the w_j are weakly contractive. If, moreover, each p_j is a Dini function (i.e., $\int_0^1 \alpha_{p_j}(t)t^{-1}\,dt < \infty$), then we call the system a *weakly contractive Dini system*. Similarly we can define the corresponding terminology for nonexpansive IFS.

In [H] Hata has studied the invariant sets of a weakly contractive IFS on X. By using the existence of fixed points for weakly contractive maps, he showed the existence of a unique nonempty compact $K \subseteq X$ invariant under the w_j's, i.e.,

$$K = \bigcup_{j=1}^m w_j(K).$$

For a multi-index $J = (j_1, \ldots, j_n)$, $1 \le j_k \le m$, let

$$w_J(x) = w_{j_1} \circ \ldots \circ w_{j_n}(x).$$

Then $\lim_{|J|\to\infty} |w_J(K)| = 0$ [H] and $K = \bigcap_{n=1}^\infty \bigcup_{|J|=n} w_J(K)$. For more general IFS, the invariant set may not be unique. However we have

PROPOSITION 2.1. *Suppose $\{w_j\}_{j=1}^m$ are continuous on the compact subset X and at least one of them is weakly contractive. Then there exists a unique smallest nonempty compact set K such that*

$$K = \bigcup_{j=1}^{m} w_j(K).$$

Moreover for any $x \in K$, the closure of $\{w_J(x) : |J| = n,\ n \in \mathbb{N}\}$ is K.

Proof. Let $\mathcal{F} = \{F : \bigcup_{j=1}^m w_j(F) \subseteq F\}$. By using the standard Zorn lemma argument, there exists a minimal compact subset K such that $K = \bigcup_{j=1}^m w_j(K)$. To show that such a K is unique, we assume without loss of generality that w_1 is weakly contractive. Let $J_n = (1, \ldots, 1)$ (n times). Then $\lim_{n\to\infty} |w_{J_n}(X)| = 0$. Let K' be another minimal compact invariant set, and let $x \in K$ and $y \in K'$. Then

$$\lim_{n\to\infty} w_{J_n}(x) = \lim_{n\to\infty} w_{J_n}(y) \in K \cap K'.$$

Hence $K \cap K' \neq \emptyset$ and $w_j(K \cap K') \subseteq K \cap K'$. The minimality implies that $K = K'$.

The last statement follows from the fact that K is the smallest invariant subset. ∎

Throughout the paper we will consider either weakly contractive IFS or IFS as in Proposition 2.1, hence the set K is uniquely defined. Furthermore we can assume that $|K| = \sup\{|x-y| : x, y \in K\} = 1$. We define an operator $T : C(K) \to C(K)$ by

$$Tf(x) = \sum_{j=1}^{m} p_j(w_j x) f(w_j x).$$

T is called the *Ruelle operator* of the system. The dual operator T^* on the measure space $M(K)$ is given by

$$T^*\mu = \sum_{j=1}^{m} p_j(\cdot)\mu \circ w_j^{-1}.$$

For $J = (j_1, \ldots, j_n)$, $1 \leq j_k \leq m$, we let

$$p_{w_J}(x) = p_{j_1}(w_{j_1} \circ w_{j_2} \circ \ldots \circ w_{j_n} x) \ldots p_{j_{n-1}}(w_{j_{n-1}} \circ w_{j_n} x) p_{j_n}(w_{j_n} x).$$

Then

$$T^n f(x) = \sum_{|J|=n} p_{w_J}(x) f(w_J x).$$

Let $\varrho = \varrho(T)$ be the spectral radius of T. Since T is a positive operator, we have $\|T^n 1\| = \|T^n\|$ and

$$\varrho = \lim_n \|T^n\|^{1/n} = \lim_n \|T^n 1\|^{1/n}.$$

PROPOSITION 2.2. *Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be an IFS with at least one w_j weakly contractive. Let T be the Ruelle operator on $C(K)$. Then*

(i) $\min_{x\in K} \varrho^{-n}T^n 1(x) \le 1 \le \max_{x\in K} \varrho^{-n}T^n 1(x)$ *for all* $n > 0$,

(ii) *if there exist* $\lambda > 0$ *and* $0 < h \in C(K)$ *such that* $Th = \lambda h$, *then* $\lambda = \varrho$ *and there exist* $A, B > 0$ *such that*

$$A \le \varrho^{-n}T^n 1(x) \le B \quad \forall n > 0.$$

Proof. We will prove the second inequality of (i); the first inequality is similar. Suppose it is not true, then there exists k such that $\|T^k 1\| < \varrho^k$. Hence

$$\varrho = (\varrho(T^k))^{1/k} \le \|T^k\|^{1/k} = \|T^k 1\|^{1/k} < \varrho,$$

which is a contradiction.

To prove (ii) we let $a_1 = \min_{x\in K} h(x)$, $a_2 = \max_{x\in K} h(x)$. Then

$$0 < \frac{a_1}{a_2} \le \frac{h(x)}{a_2} = \frac{\lambda^{-n}}{a_2}T^n h(x) \le \lambda^{-n}T^n 1(x) = \lambda^{-n}\|T^n\|.$$

Similarly we can show that $\lambda^{-n}\|T^n\| \le a_2/a_1$. Hence $\varrho = \lim_{n\to\infty} \|T^n\|^{1/n} = \lambda$. ∎

We call the operator $T : C(K) \to C(K)$ *irreducible* if for any nontrivial, nonnegative $f \in C(K)$ and for any $x \in K$, there exists $n > 0$ such that $T^n f(x) > 0$.

PROPOSITION 2.3. *Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be an IFS with at least one w_j weakly contractive. Then the Ruelle operator T is irreducible and*

$$\dim\{h \in C(K) : Th = \varrho h,\ h \ge 0\} \le 1;$$

if $h \ge 0$ is a ϱ-eigenfunction of T, then $h > 0$.

Proof. For any given $f \in C(K)$ with $f \ge 0$ and $f \not\equiv 0$, define $V = \{x \in K : f(x) > 0\}$. For any $x \in K$, by Proposition 2.1, there exists J_0 such that $w_{J_0}(x) \in V$. Let $n_0 = |J_0|$. Then

$$T^{n_0} f(x) = \sum_{|J|=n_0} p_{w_J}(x) f(w_J x) \ge p_{w_{J_0}}(x) f(w_{J_0} x) > 0.$$

This proves that T is irreducible.

For the dimension of the eigensubspace, we suppose that there exist two linearly independent strictly positive ϱ-eigenfunctions $h_1, h_2 \in C(K)$. Without loss of generality we assume that $0 < h_1 \le h_2$ and $h_1(x_0) = h_2(x_0)$ for some $x_0 \in K$. Then $h = h_2 - h_1$ (≥ 0) is a ϱ-eigenfunction of T and $h(x_0) = 0$. It follows that $T^n h(x_0) = \varrho^n h(x_0) = 0$, which contradicts the irreducibility of T. Hence the dimension of the ϱ-eigensubspace is at most 1.

The strict positivity of h follows directly from the irreducibility of T. ∎

With an iterated function system, one frequently associates a shift transformation on a symbolic space through conjugation. By a *symbolic space* we mean the infinite product space $\Sigma = \{1, \ldots, m\}^{\mathbb{N}}$. For $\sigma = (\sigma_n) \in \Sigma$, we write $\sigma|^k = (\sigma_1, \ldots, \sigma_k)$ and $\sigma|_k = (\sigma_{k+1}, \sigma_{k+2}, \ldots)$. The shift transformation on Σ is defined by $\theta(\sigma) = \sigma|_1$. We define the distance of $\sigma, \sigma' \in \Sigma$ as $d(\sigma, \sigma') = e^{-n(\sigma,\sigma')}$ where $n(\sigma, \sigma')$ is the largest n such that $\sigma|^n = \sigma'|^n$. It follows that the cylinder set $I_n(\sigma)$ is the ball of radius e^{-n} with center at σ. Define

$$u_j : \Sigma \to \Sigma \quad \text{by} \quad u_j\sigma = j\sigma, \quad 1 \le j \le m.$$

Then $\theta^{-1}(\sigma) = \{u_j(\sigma)\}$. The system $(\Sigma, \{u_j\}, \{q_j\})$ with an arbitrary choice of q_j is called a *symbolic system.* The u_j's are clearly contractive maps with contractive ratio e^{-1}. With suitably defined weights q_j, this symbolic system becomes a prototype for a general system. For our case we define $q : \Sigma \to \mathbb{R}^+$ by $q(\sigma) = q_j(\sigma) = p_j(\pi(\sigma))$ if $\sigma \in u_j(\Sigma)$ where π is defined in the next proposition. Let ν be the eigenmeasure of the Ruelle operator on the system $(\Sigma, \{u_j\}, q)$. The following establishes the "semiconjugacy" of a weakly contractive system and a symbolic system.

PROPOSITION 2.4. *Let* $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ *be a weakly contractive system. Let* $y \in K$ *be fixed and let* $\pi : \Sigma \to K$ *be defined by*

$$\pi(\sigma) = \lim_{n\to\infty} w_{\sigma|^n}(y) = \lim_{n\to\infty} w_{\sigma_1} \cdots w_{\sigma_n}(y).$$

(i) *The limit exists and is independent of* $y \in K$. *The mapping* π *is continuous and onto, and satisfies* $\pi \circ u_j = w_j \circ \pi$, $1 \le j \le m$.

(ii) *Let* μ *be the image of* ν *under* π. *Then* $T^*\mu = \varrho\mu$.

Proof. (i) is proved in [H]. The proof of (ii) is the same as in [FL, Proposition 1.3]. ∎

The proposition establishes the following commuting diagram:

$$\begin{tikzcd} \Sigma \arrow[r, shift left, "\{u_j\}"] \arrow[d, "\pi"'] & \Sigma \arrow[l, shift left, "\theta"] \arrow[d, "\pi"] \\ K \arrow[r, "\{w_j\}"] & K \end{tikzcd}$$

The classical symbolic system is the one with a positive Hölder continuous q; it has been studied in great detail in the literature (e.g., [B]) and the Hölder continuity has been extended to Dini continuity by Walters [W] and Fan [F1]. In [FL] it is proved that if $(X, \{w_j\}, \{p_j\})$ is a contractive Dini system, then it can be lifted to the symbolic system by the above semiconjugacy (π is not necessarily one-to-one) and the corresponding q remains a Dini function. Hence much of the eigenfunction properties of the Ruelle operator can be reduced to the known results on the symbolic space. For the present weakly contractive case, it is not possible to lift the Dini system to a Dini system on Σ in general. Nevertheless for some special cases we can still obtain such a correspondence. We will consider such a case in the following:

LEMMA 2.5. *For $\alpha > 0$, let $\phi(t) = t(1 - t^\alpha)$, $0 < t < 1$. Then*

$$\phi^n(t) = \phi \circ \ldots \circ \phi(t) = \frac{t}{(1 + n\alpha t)^{1/\alpha}}(1 + o_n(t))$$

where $\lim_{n\to\infty} o_n(t) = 0$ *for* $t \in [0, 1]$ *and* $\lim_{t\to 0} o_n(t) = 0$ *for each* $n > 0$.

Proof. Note that $(1-t)^\alpha = 1-\alpha t+o(t)$ where $o(t) < t$ and $\lim_{t\to 0} o(t)/t = 0$. Then

$$G(x) := (\phi(x^{-1/\alpha}))^{-\alpha} = x + \alpha + x \cdot o(x^{-1}), \qquad x \geq 1.$$

Let $G^1(x) = G(x)$, and inductively let

$$G^n(x) = x + n\alpha + R_n(x), \qquad x \geq 1,$$

where $R_n(x) = \sum_{k=0}^{n-1} G^k(x) \cdot o((G^k(x))^{-1})$. It follows that

$$\phi^n(t) = (G^n(t^{-\alpha}))^{-1/\alpha} = \frac{t}{(1 + n\alpha t)^{1/\alpha}}\left(1 - \frac{t^\alpha R_n(t^{-\alpha})}{1 + n\alpha t + t^\alpha R_n(t^{-\alpha})}\right)^{1/\alpha}.$$

Note that $G^k(x) \cdot o((G^k(x))^{-1}) \to 0$ as $k \to \infty$ and as $x \to \infty$. This implies the lemma. ∎

PROPOSITION 2.6. *For an IFS $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$, suppose $\alpha_{w_j}(t) \leq t(1 - t^\alpha)$ for $0 < t < 1$, and $\alpha_{\log p_j}(t) = O(t^\beta)$ for some $0 < \alpha < \beta \leq 1$. Then the associated symbolic dynamical system is a Dini system, i.e., q is a Dini function.*

Proof. For a multi-index J, we can define, analogously to p_{w_J},

$$q_{u_J}(\sigma) = q(u_{j_1} \ldots u_{j_n}(\sigma)) \ldots q(u_{j_{n-1}} u_{j_n}(\sigma)) q(u_{j_n}(\sigma)).$$

Then for $x = \pi(\sigma)$, we have $q_{u_J}(\sigma) = p_{w_J}(x)$. Note that $\sigma = u_{\sigma|^n}(\theta^n\sigma)$. Then for σ and σ' in Σ such that $\sigma|^n = \sigma'|^n$,

$$\begin{aligned}
&|\log q(\sigma) - \log q(\sigma')| \\
&\quad = |\log p_{\sigma_1}(\pi \circ u_{\sigma|^n}(\theta^n\sigma)) - \log p_{\sigma_1}(\pi \circ u_{\sigma|^n}(\theta^n\sigma'))| \\
&\quad = |\log p_{\sigma_1}(w_{\sigma|^n} \circ \pi(\theta^n\sigma)) - \log p_{\sigma_1}(w_{\sigma|^n} \circ \pi(\theta^n\sigma'))| \\
&\quad \leq \sup_{x,y} |\log p_{\sigma_1}(w_{\sigma|^n}(x)) - \log p_{\sigma_1}(w_{\sigma|^n}(y))| \\
&\quad \leq \max_{j,|J|=n} \alpha_{\log p_j}(\alpha_{w_J}(1)) \leq C((1 + n\alpha)^{-1/\alpha})^\beta \qquad \text{(by Lemma 2.5)}
\end{aligned}$$

for some $C > 0$. It follows that

$$\sum_{n=1}^{\infty} \alpha_{\log q}(e^{-n}) \leq \sum_{n=1}^{\infty} \frac{C}{(1+n\alpha)^{\beta/\alpha}} < \infty,$$

and $\log q$ is Dini continuous. ∎

The eigenproblem for the Ruelle operator is well understood once the system is semiconjugate to a symbolic Dini system. The reader is referred to [FL, Theorem 1.1] for the details. In the remaining sections we will consider the more general case without recourse to the symbolic system.

To conclude this section we make a digression on the Ruelle operator in the setting of Rényi [Re], Gel'fond [G] and Parry [P]. Let $g : [0,1] \to [0,1]$ be a piecewise continuously differentiable function with $|g'(x)| \geq 1$. By the ergodic theorem, there exists an invariant measure ν such that $\nu = \nu \circ g^{-1}$. In order for ν to be absolutely continuous with respect to the Lebesgue measure, it is necessary that there exists an $h \in L^1[0,1]$ such that

$$h(x) = \sum_{y \in g^{-1}(x)} (g'(y))^{-1} h(y).$$

To put it into our notation, we let $\{w_j\}_{j=1}^m$ be the m branches of g^{-1} and let $p_j(w_j x) = (g'(y))^{-1}$ for $y = w_j(x)$. Assuming that all the w_j are defined on $[0,1]$, we can define the Ruelle operator T. Then h is a nonnegative 1-eigenfunction of T (on $L^1[0,1]$).

If the w_j's are contractive (i.e., g is hyperbolic) and $\log|w_j'(\cdot)|$, $1 \leq j \leq m$, are Dini functions, then the above h always exists (see [FL]). However it is not the case if the w_j's are weakly contractive. We consider the following example by Lasota and Yorke [LY]: let

$$g(x) = \begin{cases} \dfrac{x}{1-x} & \text{if } x \in [0,1/2], \\ 2x-1 & \text{if } x \in (1/2,1]. \end{cases}$$

The two branches of g^{-1} are given by $w_1(x) = x/(1+x)$, $w_2(x) = 1/2 + x/2$. Let $p_1(w_1 x) = 1/(1+x)^2$ and $p_2(w_2 x) = 1/2$. Then

$$Tf(x) = \sum_{y \in g^{-1}(x)} \frac{f(y)}{g'(y)} = \frac{1}{(1+x)^2} f\left(\frac{x}{1+x}\right) + \frac{1}{2} f\left(\frac{1}{2} + \frac{x}{2}\right).$$

It was proved in [LY] that there is no L^1-solution. Here we consider $T : C[0,1] \to C[0,1]$. Then the spectral radius of T is 1 (the proof will be given after Corollary 4.9). We can easily see that there is no positive continuous 1-eigenfunction. Indeed, if h is such a function in $C[0,1]$, then

$$h(0) = h(0) + \frac{1}{2} h\left(\frac{1}{2}\right),$$

which is impossible.

If we modify the above operator T on $C(K)$ to

$$Tf(x) = \frac{1}{(1+x)^2} f\left(\frac{x}{1+x^\alpha}\right) + \frac{1}{2} f\left(\frac{1}{2} + \frac{x}{2}\right)$$

for some $0 < \alpha < 1$, then it is easy to see that $\alpha_{w_1}(t) \le t(1 - t^{(1+\alpha)/2})$ and $\alpha_{p_1}(t) = O(t)$. Proposition 2.6 implies that the ϱ-eigenfunction h exists. However the explicit value of ϱ is difficult to find. A numerical algorithm was considered by Strichartz *et al.* [STZ].

3. Perron–Frobenius property. We first give a basic criterion for the existence of an eigenfunction corresponding to the spectral radius ϱ.

PROPOSITION 3.1. *Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be an IFS with at least one w_j weakly contractive. Suppose that*

(i) *there exist $A, B > 0$ such that $A \le \varrho^{-n} T^n 1(x) \le B$ for any $x \in K$ and $n > 0$,*

(ii) *for any $f \in C(K)$, $\{\varrho^{-n} T^n f\}_{n=1}^\infty$ is an equicontinuous sequence.*

Then there exists a unique $0 < h \in C(K)$ and a unique probability measure $\mu \in M(K)$ such that

$$Th = \varrho h, \qquad T^* \mu = \varrho \mu, \qquad \langle \mu, h \rangle = 1.$$

Moreover, for every $f \in C(K)$, $\varrho^{-n} T^n f$ converges to $\langle \mu, f \rangle h$ in the supremum norm, and for every $\xi \in M(K)$, $\varrho^{-n} T^{*n} \xi$ converges weakly to $\langle \xi, h \rangle \mu$.

Proof. The proof is modified from [W, Theorem 3.1] on the symbolic space. We include the details here for completeness. Let

$$f_n(x) = \frac{1}{n} \sum_{i=0}^{n-1} \varrho^{-i} T^i 1(x).$$

Then $\{f_n\}_{n=1}^\infty$ is bounded by A and B and is an equicontinuous subset of $C(K)$. By the Arzelà–Ascoli theorem, we can assume that there exists an $h \in C(K)$ such that $\lim_n \|f_n - h\| = 0$. Hence

$$\|Th - \varrho h\| = \lim_n \|Tf_n - \varrho f_n\| \le \lim_n \frac{\varrho}{n} \|1 - \varrho^{-n} T^n 1\| \le \lim_n \frac{\varrho}{n}(1 + B) = 0,$$

i.e., $Th = \varrho h$ and also $h \ge A > 0$. We let

$$q_j(x) = \frac{p_j(w_j x) h(w_j x)}{\varrho h(x)}$$

and define a "normalized" operator $L : C(K) \to C(K)$ by

$$Lf(x) = \sum_{j=1}^m q_j(x) f(w_j x).$$

Note that $\sum_{j=1}^m q_j(x) = 1$ and the 1 function is a 1-eigenfunction of L.

For $f \in C(K)$, we have $T^n f = \varrho^n h L^n(f \cdot h^{-1})$, hence $\{L^n f\}_{n=1}^{\infty}$ is a bounded equicontinuous sequence in $C(K)$. We know from the Arzelà–Ascoli theorem that there exists $\widetilde{f} \in C(K)$ and a subsequence $\{L^{n_i} f\}_{i=1}^{\infty}$ such that $\lim_i \|L^{n_i} f - \widetilde{f}\| = 0$.

We claim that $\widetilde{f}$ is a constant function and $\lim_n \|L^n f - \widetilde{f}\| = 0$. For this we let $\tau(g) = \min_x g(x)$. Since $\sum_{j=1}^{m} q_j(x) = 1$, it is easy to see that $\tau(\widetilde{f}) \le \tau(L\widetilde{f})$ and

$$\tau(f) \le \tau(Lf) \le \ldots \le \tau(\widetilde{f}). \tag{3.1}$$

By taking the limit, we have $\tau(L\widetilde{f}) \le \tau(\widetilde{f})$ and hence equality holds. For any $n > 0$, we choose $x_n \in K$ satisfying $L^n \widetilde{f}(x_n) = \tau(L^n \widetilde{f}) = \tau(\widetilde{f})$. Then $\sum_{|J|=n} q_{w_J}(x_n) = 1$ implies that $\widetilde{f}(w_J x_n) = \tau(\widetilde{f})$ for every J with $|J| = n$. Similarly there exists $y_n \in K$ such that $\widetilde{f}(w_J y_n) = \eta(\widetilde{f}) := \max_x \widetilde{f}(x)$ for every J with $|J| = n$. As in Proposition 2.1, we assume w_1 is weakly contractive and let $J_n = (1, \ldots, 1)$ with $|J_n| = n$. Then $z := \lim_n w_{J_n}(x_n) = \lim_n w_{J_n}(y_n) \in K$. Hence

$$\tau(\widetilde{f}) = \lim_n \widetilde{f}(w_{J_n} x_n) = \widetilde{f}(z) = \lim_n \widetilde{f}(w_{J_n} y_n) = \eta(\widetilde{f}).$$

Thus $\widetilde{f}(x) \equiv \tau(\widetilde{f})$ is a constant function. By (3.1) and the dual version for $\eta(\widetilde{f})$, we have $\lim_n \|L^n f - \widetilde{f}\| = 0$.

In particular, by taking $f = h^{-1}$, we see that $L^n(h^{-1})$ converges uniformly, and then $\varrho^{-n} T^n 1$ converges uniformly. Since the average of $\varrho^{-n} T^n 1$ converges to h as at the beginning of the proof, we have $\lim_n \|\varrho^{-n} T^n 1 - h\| = 0$, and hence $\lim_n \|L^n(h^{-1}) - 1\| = 0$.

Now we define a function $v : C(K) \to \mathbb{R}$ by $\langle v, f \rangle = \tau(\widetilde{f})$ $(= \widetilde{f}(x)$ for any fixed $x \in K)$. Then v is a bounded linear functional on $C(K)$, $\langle v, 1 \rangle = 1$, $\langle v, h^{-1} \rangle = 1$. From

$$\langle v, Lf \rangle = \tau(L\widetilde{f}) = \tau(\widetilde{f}),$$

we have $L^* v = v$. Let $\mu : C(K) \to \mathbb{R}$ be defined by $\langle \mu, f \rangle = \langle v, f h^{-1} \rangle$. Then $\langle \mu, 1 \rangle = \langle v, h^{-1} \rangle = 1$ and μ is a probability measure. It is easy to see that $T^* \mu = \varrho \mu$ and $\langle \mu, h \rangle = \langle v, 1 \rangle = 1$. Hence for any $f \in C(K)$, $\varrho^{-n} T^n f$ converges to $\langle \mu, f \rangle h$ in the supremum norm. Also it follows that for every $\xi \in M(K)$, $\varrho^{-n} T^{*n} \xi$ converges weakly to $\langle \xi, h \rangle \mu$.

The uniqueness of the eigenfunction follows from Proposition 2.3. For the uniqueness of the eigenmeasure, we observe that if $\sigma \in M(K)$ satisfies $T^* \sigma = \varrho \sigma$ and $\langle \sigma, h \rangle = 1$, then for any $f \in C(K)$,

$$\langle \sigma, f \rangle = \lim_n \langle \varrho^{-n} T^{*n} \sigma, f \rangle = \lim_n \langle \sigma, \varrho^{-n} T^n f \rangle = \langle \sigma, \langle \mu, f \rangle h \rangle = \langle \mu, f \rangle.$$

Hence $\sigma = \mu$. ∎

DEFINITION 3.2. The Ruelle operator T on $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ is said to have the *PF-property* (Perron–Frobenius) if there exists a unique $0 < h \in C(K)$ and a unique probability measure $\mu \in M(K)$ such that

$$Th = \varrho h, \qquad T^*\mu = \varrho\mu, \qquad \langle \mu, h\rangle = 1,$$

and for every $f \in C(K)$, $\varrho^{-n}T^n f$ converges to $\langle \mu, f\rangle h$ in the supremum norm.

It is known that symbolic Dini systems and contractive Dini systems have the PF-property ([F1], [FL] and [W]). Proposition 2.6 shows that some weak contractive systems also have this property. In the next two sections, we will consider other systems under the framework of Proposition 3.1. The basic method is to construct an auxiliary function Φ to check the equicontinuity of $\{\varrho^{-n}T^n f\}_{n=1}^\infty$ in Proposition 3.1. We summarize it in the following two lemmas.

LEMMA 3.3. *Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be an IFS with at least one w_j weakly contractive. Suppose that*

(i) $\sup_n \|\varrho^{-n}T^n\| < \infty$,

(ii) *there exists a dense subset D of $C^+(K) := \{f \in C(K) : f > 0\}$ such that for each $f \in D$, there exists a continuous function Φ (depending on f) on $[0,1]$ with $\Phi(0) = 0$ such that*

$$0 < T^n f(x) \le T^n f(y) e^{\Phi(|x-y|)} \qquad \forall x, y \in K,\ \forall n \ge 0.$$

Then for each $f \in C(K)$, $\{\varrho^{-n}T^n f\}_{n=1}^\infty$ is a bounded equicontinuous sequence.

Proof. Let $f \in D$, $g \in C(K)$. For any $x, y \in K$ and $n > 0$,

$$\begin{aligned}
&|\varrho^{-n}T^n g(x) - \varrho^{-n}T^n g(y)| \\
&\qquad \le \|\varrho^{-n}T^n f\| \left| 1 - \frac{T^n f(y)}{T^n f(x)} \right| + 2\|\varrho^{-n}T^n\| \cdot \|f - g\| \\
&\qquad \le B(\|f\|(e^{\Phi(|x-y|)} - 1) + 2\|f - g\|)
\end{aligned}$$

where $B = \sup_n \|\varrho^{-n}T^n\|$. By the assumptions on D and Φ, we can show that for each $f \in C^+(K)$, $\{\varrho^{-n}T^n f\}_{n=1}^\infty$ is a bounded equicontinuous subset of $C(K)$.

For $f \in C(K)$, we can choose $a > 0$ such that $f + a > 0$. Then $\{\varrho^{-n}T^n(f + a)\}_{n=1}^\infty$ and $\{\varrho^{-n}T^n a\}_{n=1}^\infty$ are bounded equicontinuous subsets of $C^+(K)$, hence $\{\varrho^{-n}T^n f\}_{n=1}^\infty$ is also a bounded equicontinuous subset of $C(K)$. ∎

The lemma will be used in Section 4. Since the spectral radius is not given *a priori*, condition (i) may not be easy to check in many cases. We present another criterion which will imply the condition. Recall that a nonempty subset F of $C(K)$ is called a *cone* if $af + bg \in F$ for any $a, b > 0$ and $f, g \in F$.

LEMMA 3.4. *Suppose an IFS satisfies* (ii) *above. Furthermore suppose that D contains a cone F closed in $C^+(K)$ such that the Φ in* (ii) *is independent of f in F and $T(F) \subseteq F$. Then T has the PF-property.*

Proof. Let $F_0 = \{f \in F : e^{-\Phi(1)} \le f \le 1\}$. Then F_0 is a bounded convex equicontinuous subset of $C(K)$. To show that F_0 is closed in $C(K)$, we observe that F is closed in $C^+(K)$, there exists a closed subset E of $C(K)$ such that $F = E \cap C^+(K)$, hence $F_0 = \{f \in E : e^{-\Phi(1)} \le f \le 1\}$ is closed in $C(K)$. It is also compact by the equicontinuity of F_0. We define $\mathcal{L} : F_0 \to C(K)$ by

$$\mathcal{L}f(x) = Tf(x)/\|Tf\|.$$

Then

$$\mathcal{L}f(x) \le \mathcal{L}f(y)e^{\Phi(|x-y|)} \qquad \forall f \in F_0.$$

This implies that

$$1 = \max_{x \in K} \mathcal{L}f(x) \le \mathcal{L}f(y)e^{\Phi(1)} \qquad \forall y \in K.$$

Consequently,

$$e^{-\Phi(1)} \le \mathcal{L}f(x) \le 1 \qquad \forall x \in K.$$

Hence $\mathcal{L}F_0 \subseteq F_0$. The Schauder fixed point theorem yields an $h \in F_0$ such that $\mathcal{L}h = h$. Then $Th = \varrho h$ where $\varrho = \|Th\|$. Condition (i) of Lemma 3.3 is hence satisfied and the PF-property follows. ∎

In the following we apply Lemma 3.4 to a weakly contractive system slightly more general than that in Proposition 2.6. We say a function $\varphi : [0,1] \to \mathbb{R}^+$ satisfies the *modulus condition* if φ is continuous, increasing, concave and $\varphi(0) = 0$. For such a φ, we see that for $0 \le t_1 < t_2$, if we let $\lambda = t_1/t_2$,

$$\frac{t_1}{t_2}\varphi(t_2) = \lambda\varphi(t_2) + (1-\lambda)\varphi(0) \le \varphi(\lambda t_2 + (1-\lambda)0) = \varphi(t_1).$$

Hence $\varphi(t)/t$ is decreasing.

THEOREM 3.5. *Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be a weakly contractive IFS satisfying*

(i) *there exist positive functions $\{\beta_j(t)\}_{j=1}^m$ on $[0,1]$ such that $\alpha_{w_j}(t) \le t(1-\beta_j(t))$,*

(ii) *there exists a Dini modulus function $\varphi(t)$ such that $\alpha_{\log p_j}(t)/\beta_j(t) \le \varphi(t)$ for each j.*

Then T has the PF-property.

Proof. Let

$$\Phi(t) = t + \int_0^t \frac{\varphi(x)}{x}\,dx, \qquad t \in [0,1].$$

Then $\Phi(t)$ is increasing and continuous on $[0,1]$ and $\Phi(0) = 0$. Hence for each $1 \le j \le m$,

$$\Phi(t) - \Phi(t(1-\beta_j(t))) \ge \int_{t(1-\beta_j(t))}^{t} \frac{\varphi(x)}{x}\,dt \ge \frac{\varphi(t)}{t}\cdot t\beta_j(t) \ge \alpha_{\log p_j}(t). \tag{3.2}$$

We will prove that T satisfies the conditions of Lemma 3.4. Let

$$D_k = \{f \in C^+(K) : f(x) \le f(y)e^{k\Phi(|x-y|)}\}.$$

Then $D := \bigcup_{k=1}^{\infty} D_k$ is dense in $C^+(K)$. For $f \in D_k$ and $x, y \in K$,

$$\begin{aligned} Tf(x) &= \sum_{j=1}^{m} p_j(w_jx)f(w_jx) \\ &\le \sum_{j=1}^{m} p_j(w_jy)f(w_jy)e^{\alpha_{\log p_j}(|x-y|)+k\Phi(|x-y|(1-\beta_j(|x-y|)))} \\ &\le \sum_{j=1}^{m} p_j(w_jy)f(w_jy)e^{k\Phi(|x-y|)} \qquad \text{(by (3.2))} \\ &\le Tf(y)e^{k\Phi(|x-y|)}. \end{aligned}$$

It follows that $TD_k \subseteq D_k$ and D is invariant for T, and condition (ii) of Lemma 3.3 is satisfied (the $k\Phi$ here corresponding to the Φ in Lemma 3.4). Note that D_1 is a closed cone of $C^+(K)$ and Φ is independent of f on D_1. Hence Lemma 3.4 implies the PF-property of T. ■

PROPOSITION 3.6. *Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be as in Theorem 3.5 and let μ be the ϱ-eigenmeasure of T^*. Suppose $\mu(K_I \cap K_J) = 0$ for all $I \ne J$ with $|I| = |J|$. Then μ has the Gibbs property, i.e., there exists $c \ge 1$ such that for any J and $x \in K$,*

$$c^{-1}\varrho^{-|J|}p_{w_J}(x) \le \mu(K_J) \le c\varrho^{-|J|}p_{w_J}(x).$$

Proof. We first claim that there exists $c \ge 1$ such that $p_{w_J}(x) \le cp_{w_J}(y)$ for all J and $x, y \in K$. Indeed, let $z_j(t) = t(1-\beta_j(t))$, $\alpha_j = \alpha_{\log p_j}$. For $J = (j_1, \ldots, j_n)$, $0 \le k \le n-1$, let $J|_k = (j_{k+1}, j_{k+2}, \ldots, j_n)$ and $a_k = z_{J|_k}(1)$. Then

$$\left|\log\frac{p_{w_J}(x)}{p_{w_J}(y)}\right| \le \sum_{k=0}^{n-1} \alpha_{j_{k+1}}(|w_{J|_k}(K)|) \le \sum_{k=0}^{n-1} \alpha_{j_{k+1}}(a_k) \le \sum_{k=0}^{n-1} \alpha_{j_{k+1}}(a_{k+1}).$$

By (3.2), we have

$$\alpha_{j_{k+1}}(a_{k+1}) \le \Phi(a_{k+1}) - \Phi(a_k).$$

Hence

$$\left|\log \frac{p_{w_J}(x)}{p_{w_J}(y)}\right| \le \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} \frac{\varphi(x)}{x}\,dx \le \int_0^1 \frac{\varphi(x)}{x}\,dx < \infty$$

and the claim follows.

To prove the Gibbs property of the invariant measure μ, we note that $T^{*n}\mu = \varrho^n \mu$. Hence by the assumption that $\mu(K_I \cap K_J) = 0$ for all $I \neq J$ with $|I| = |J|$, we have

$$\begin{aligned}\mu(K_J) &= \langle \mu, 1_{K_J}\rangle = \langle \varrho^{-n} T^{*n}\mu, 1_{K_J}\rangle = \langle \mu, \varrho^{-n} T^n 1_{K_J}\rangle \\ &= \Big\langle \mu, \varrho^{-n} \sum_{|I|=n} p_{w_I}(\cdot) 1_{K_J}(w_I(\cdot))\Big\rangle = \langle \mu, \varrho^{-n} p_{w_J}(\cdot)\rangle.\end{aligned}$$

It follows from the claim that there exists $c \ge 1$ such that

$$c^{-1}\varrho^{-|J|} p_{w_J}(x) \le \mu(K_J) \le c\varrho^{-|J|} p_{w_J}(x). \ \blacksquare$$

Note that conditions (i), (ii) of Theorem 3.5 are satisfied if w_j's are contractive and p_j's are Dini continuous.

The condition $\mu(K_I \cap K_J) = 0$ for all $I \neq J$ with $|I| = |J|$ is closely related to the open set condition. It has been discussed in detail in [FL] and we will make some remarks on it at the end of the paper.

4. Some sufficient conditions. Throughout this section we will consider nonexpansive Dini systems, and apply Proposition 3.1 and Lemma 3.3 to study the eigenproblem for the Ruelle operator. In conjunction with the "bounded distortion property" of $T^n f$ in Lemma 3.3, we see in the next lemma that the Dini condition on the p_j's also implies a property of similar nature. Recall that an equivalent condition for $p(x)$ to be Dini continuous is $\sum_n \alpha_p(\theta^n) < \infty$ for $0 < \theta < 1$.

LEMMA 4.1. *Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be an IFS such that the p_j's are Dini functions. Let $\alpha(t) = \max_j \alpha_{\log p_j}(t)$, $0 < \theta < 1$ and $a = \sum_{n=0}^{\infty} \alpha(\theta^n)$. If $J = (j_1, \ldots, j_n)$ satisfies $|w_{j_i \ldots j_n}(K)| \le \theta^{n-i}$ for all $1 \le i \le n$. Then*

$$p_{w_J}(x) \le e^a p_{w_J}(y) \quad \forall x, y \in K.$$

Proof. The inequality follows from the estimate

$$\left|\log \frac{p_{w_J}(x)}{p_{w_J}(y)}\right| \le \sum_{i=1}^n |\log p_{j_i}(w_{j_i \ldots j_n} x) - \log p_{j_i}(w_{j_i \ldots j_n} y)| \le \sum_{i=1}^n \alpha(\theta^{n-i}) \le a. \ \blacksquare$$

THEOREM 4.2. *Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be a nonexpansive Dini system. Suppose that*

(i) $\min_{1\le j\le m}\sup_{x\ne y}|w_j(x)-w_j(y)|/|x-y| = r < 1$,

(ii) *there exist constants* $A, B > 0$ *such that* $A \le \varrho^{-n}T^n 1(x) \le B$ *for any* $x \in K$ *and* $n > 0$.

Then T *has the PF-property.*

Proof. Let $D = \{f \in C^+(K) : f(x) \le f(y)e^{c|x-y|}$ for some $c > 0\}$. Then D is dense in $C^+(K)$. For any $f \in D$, $c_1^{-1} \le f(x) \le c_1$ for some $c_1 > 0$, and by assumption (ii),

$$Ac_1^{-1} \le \varrho^{-n}T^n f(x) \le Bc_1.$$

Combining this with the strict positivity of p_j, it is straightforward to show that

$$0 < b := \inf_{n\ge1}\min_{x,j}\frac{p_j(w_jx)T^{n-1}f(w_jx)}{T^nf(x)} < 1. \tag{4.1}$$

For $t > 0$, let $\alpha(t) = \max\{t, \max_j \alpha_{\log p_j}(t)\}$. Then $\alpha(t)$ satisfies the Dini condition. Choose $k \ge 1$ large enough such that $kb \ge 1$ and define

$$\Phi(t) = \frac{k+c}{1-r}\int_0^t \frac{\alpha(x/r)}{x}\,dx.$$

By a direct calculation, we have

$$ct \le \Phi(t), \qquad k\alpha(t) + \Phi(rt) \le \Phi(t), \tag{4.2}$$

and hence $f(x) \le f(y)e^{\Phi(|x-y|)}$. We will prove that for any $x, y \in K$ and $n > 0$,

$$T^nf(x) \le T^nf(y)e^{\Phi(|x-y|)}.$$

Indeed,

$$\begin{aligned} Tf(y) &= Tf(x)\sum_{j=1}^m \frac{p_j(w_jy)f(w_jy)}{Tf(x)} \\ &\ge Tf(x)\sum_{j=1}^m \frac{p_j(w_jx)f(w_jx)}{Tf(x)}e^{-\alpha(|x-y|)-\alpha_{\log f}(|w_jx-w_jy|)} \\ &\ge Tf(x)e^{-\alpha(t)-S} \qquad \text{(by the convexity of } e^x) \end{aligned}$$

where $t = |x-y|$ and

$$S = \sum_{j=1}^m \frac{p_j(w_jx)f(w_jx)}{Tf(x)}\alpha_{\log f}(|w_jx-w_jy|).$$

From (i) we can assume that $|w_1(x)-w_1(y)|/t \le r$; then

$$\alpha_{\log f}(|w_1x-w_1y|) \le \Phi(rt),$$

and by the nonexpansiveness of $w_j, 2 \le j \le m$, we have

$$\alpha_{\log f}(|w_j x - w_j y|) \le \Phi(t).$$

We continue the above estimate on S:

$$\begin{aligned} S &\le \frac{p_1(w_1 x) f(w_1 x)}{Tf(x)}(\Phi(rt) - \Phi(t)) + \Phi(t) \\ &\le -bk\alpha(t) + \Phi(t) \qquad \text{(by (4.1), (4.2))} \\ &\le -\alpha(t) + \Phi(t). \end{aligned}$$

Hence $Tf(x) \le Tf(y)e^{\Phi(|x-y|)}$. Inductively we prove that

$$T^n f(x) \le T^n f(y) e^{\Phi(|x-y|)}.$$

The PF-property now follows from Lemma 3.3 and Proposition 3.1. ∎

COROLLARY 4.3. *Suppose $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ is a nonexpansive Dini system. If one of the w_j is contractive and $\sum_{j=1}^m p_j(x) = 1$, then T has the PF-property.*

Proof. The equality $\sum_{j=1}^m p_j(x) = 1$ implies that $\varrho = 1$, and the conditions in Theorem 4.2 are satisfied. ∎

We define $r_j = \sup_{x \ne y} |w_j(x) - w_j(y)|/|x-y|$, $r_J = r_{j_1} \ldots r_{j_n}$, and $R_J = \sup_{x\ne y} |w_J(x) - w_J(y)|/|x-y|$. As a consequence of Theorem 4.2, we have

THEOREM 4.4. *Suppose $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ is a nonexpansive Dini system. If there exists k such that*

$$\Big\| \sum_{|J|=k} p_{w_J}(\cdot) R_J \Big\| < \varrho^k, \tag{4.3}$$

then T has the PF-property.

Proof. Since T^k having the PF-property implies that T has the PF-property, we may assume $k = 1$ in the hypothesis on R_J, so that (4.3) is reduced to

$$\Big\| \sum_{j=1}^m p_j \circ w_j(\cdot) r_j \Big\| < \varrho. \tag{4.4}$$

By Proposition 2.2(i), noting that

$$\varrho = \lim_n \|T^n 1\|^{1/n} = \lim_n \Big\| \sum_{|J|=n} p_{w_J}(\cdot) \Big\|^{1/n},$$

it is easy to see that at least one of the r_j is less than 1, i.e., w_j is contractive. Without loss of generality, we assume that w_1 is such a map, hence condition (i) of Theorem 4.2 is satisfied. We need to show that condition (ii) of Theorem 4.2 is also satisfied, i.e., there exist $A, B > 0$ such that

$$A \leq \varrho^{-n} \sum_{|J|=n} p_{w_J}(x) \leq B.$$

By (4.4) we can find $0 < \eta < 1$ such that $\max_{x \in K} \sum_{j=1}^{m} p_j(w_j x) r_j < \eta\varrho$ and by induction,

$$\max_{x \in K} \sum_{|J|=n} p_{w_J}(x) r_J \leq (\eta\varrho)^n \qquad \forall n > 0. \tag{4.5}$$

For $J = (j_1, \dots, j_n)$ and $0 \leq k < l \leq n$, let $J|_k^l = (j_{k+1}, j_{k+2}, \dots, j_l)$. Choose θ such that $0 < \eta < \theta < 1$ and let

$$\Omega(n,k) = \{J : |J| = n,\ k \text{ smallest with } r_{J|_k^n} \geq \theta^{n-k}\}, \qquad 1 \leq k < n,$$
$$\Omega(n,n) = \{J : |J| = n,\ r_{J|_k^n} < \theta^{n-k}\ \forall 0 \leq k < n\}.$$

Then $\{J : |J| = n\} = \bigcup_{k=0}^{n} \Omega(n,k)$. By (4.5), we have

$$\varrho^{-n} \sum_{J \in \Omega(n,0)} p_{w_J}(x) \leq \left(\frac{\eta}{\theta}\right)^n. \tag{4.6}$$

(We use $|K|=1$ here.) Let $\alpha(t) = \max_{1 \leq j \leq m} \alpha_{\log p_j}(t)$ and $a := \sum_{k=0}^{\infty} \alpha(\theta^k)$. Then a is finite because the $\log p_i$'s are Dini functions. For any $n > 0$, we can make use of Proposition 2.2(i) to find $x_n \in K$ such that

$$\varrho^{-n} \sum_{|J|=n} p_{w_J}(x_n) \leq 1. \tag{4.7}$$

For any $J = (j_1, \dots, j_n) \in \Omega(n,k)$ and for any $0 \leq i < k$, since $r_{J|_k^n} \geq \theta^{n-k}$ and $r_{J|_i^n} = r_{J|_i^k} \cdot r_{J|_k^n} < \theta^{n-i}$, we have

$$r_{J|_i^k} = \frac{r_{J|_i^k} \cdot r_{J|_k^n}}{r_{J|_k^n}} < \theta^{k-i}.$$

By Lemma 4.1, we have $p_{w_{J|_0^k}}(y) \leq e^a p_{w_{J|_0^k}}(z)$. Hence

$$p_{w_J}(x) = p_{w_{J|_0^k}}(w_{J|_k^n} x) p_{w_{J|_k^n}}(x) \leq e^a p_{w_{J|_0^k}}(x_k) p_{w_{J|_k^n}}(x). \tag{4.8}$$

It follows that

$$\begin{aligned} \varrho^{-n} \sum_{|J|=n} p_{w_J}(x) &= \varrho^{-n} \sum_{k=0}^{n} \sum_{J \in \Omega(n,k)} p_{w_J}(x) \\ &\leq \varrho^{-n} \sum_{k=0}^{n} \sum_{J \in \Omega(n,k)} e^a p_{w_{J|_0^k}}(x_k) p_{w_{J|_k^n}}(x) \qquad \text{(by (4.8))} \end{aligned} \tag{4.9}$$

$$\le e^a \sum_{k=0}^{n} \Big(\varrho^{-k} \sum_{|J'|=k} p_{w_{J'}}(x_k)\Big)\Big(\varrho^{-(n-k)} \sum_{J''\in\Omega(n-k,0)} p_{w_{J''}}(x)\Big)$$

$$\le e^a \sum_{k=0}^{n} 1\cdot(\eta/\theta)^{n-k} \qquad (\text{by } (4.6), (4.7)).$$

The last term is bounded by $e^a \sum_{k=0}^{\infty}(\eta/\theta)^k =: B_1$. This yields the upper estimate.

For the lower estimate, we let $\alpha_J = \sum_{k=0}^{n} \alpha(|w_{J|_k^n}(K)|)$. Then it is easy to see that $\alpha_J \le a + (n-k)\alpha(1)$ for any $J \in \Omega(n,k)$. Proposition 2.2(i) and (4.9) imply that for any $n > 0$, there exists $y_n \in K$ such that

$$1 \le C_n := \varrho^{-n} \sum_{|J|=n} p_{w_J}(y_n) \le B_1.$$

Using the same argument as for (4.9), we have

$$\varrho^{-n} \sum_{|J|=n} p_{w_J}(y_n)\alpha_J = \varrho^{-n} \sum_{k=0}^{n} \sum_{J\in\Omega(n,k)} p_{w_J}(y_n)\alpha_J$$

$$\le \varrho^{-n} \sum_{k=0}^{n} (a + (n-k)\alpha(1)) \sum_{J\in\Omega(n,k)} p_{w_J}(y_n) \le B_2.$$

By the definition of α_J and the convexity of e^x, we have

$$\varrho^{-n} \sum_{|J|=n} p_{w_J}(x) \ge \varrho^{-n} \sum_{|J|=n} p_{w_J}(y_n) e^{-\alpha_J} \ge \frac{\varrho^{-n}}{C_n} \sum_{|J|=n} p_{w_J}(y_n) e^{-\alpha_J}$$

$$\ge e^{-(\varrho^{-n}\sum_{|J|=n} p_{w_J}(y_n)\alpha_J)/C_n} \ge e^{-B_2}.$$

This completes the proof. ■

It is obvious that if $\{w_j\}_{j=1}^m$ are contractive maps, then the condition in the theorem is trivially satisfied. In general, it is difficult to determine the spectral radius ϱ of T. A simple lower bound on ϱ is

$$\min_{x\in K} \sum_{j=1}^{m} p_j(w_j x) \le \varrho. \tag{4.10}$$

By using this we have

COROLLARY 4.5. *Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be a nonexpansive Dini system. If*

$$\Big\| \sum_{j=1}^{m} p_j \circ w_j(\cdot) r_j \Big\| < \min_{x\in K} \sum_{j=1}^{m} p_j(w_j x),$$

then T has the PF-property.

We remark that the expression $\sum_{|J|=k} p_{w_J}(\cdot)R_J$ in Theorem 4.4 is not so easy to handle for $k > 1$. In view of $\sum_{|J|=k} p_{w_J}(\cdot)R_J \le \sum_{|J|=k} p_{w_J}(\cdot)r_J$, a slightly stronger condition is $\| \sum_{|J|=k} p_{w_J}(\cdot)r_J\| < \varrho^k$. The next Theorem 4.7 offers a better way to check the condition. First we will prove a lemma.

LEMMA 4.6. *For $0 < t < 1$ and $n > 0$, let*

$$\mathcal{J}_n(t) = \{J = (j_1, \ldots, j_n) : j_i = 1, 2,\ \#\{j_i = 1\} \le nt\}.$$

Then for any $q > 1$, there exists $t_0 > 0$ such that $\#\mathcal{J}_n(t) < q^n$ for $0 < t \le t_0$ and $n > 0$.

Proof. By the binomial theorem, we have $\#\mathcal{J}_n(t) = \sum_{k\le nt} \binom{n}{k}$. Since for $0 < y < \min\{1, q-1\}$ we have

$$(1+y)^n \ge \sum_{k\le nt} \binom{n}{k} y^k \ge y^{nt} \sum_{k\le nt} \binom{n}{k},$$

it follows that

$$\#\mathcal{J}_n(t) \le \left(\frac{1+y}{y^t}\right)^n.$$

As $g(t) = (1+y)/y^t$ is continuous and increasing on $[0, 1]$ and $g(0) = 1 + y < q$, there exists $t_0 > 0$ such that for $0 < t \le t_0$,

$$\#\mathcal{J}_n(t) \le (g(t))^n \le (g(t_0))^n < q^n. \ \blacksquare$$

THEOREM 4.7. *Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be a nonexpansive Dini system and suppose that $w_1, \ldots, w_l$ are contractive for some $1 \le l \le m$. Then*

$$\left\| \sum_{j=l+1}^{m} p_j \circ w_j \right\| < \varrho$$

implies that there exists $n > 0$ such that $\| \sum_{|J|=n} p_{w_J}(\cdot)r_J\| < \varrho^n$.

As a direct consequence of Theorem 4.4, the above Ruelle operator has the PF-property.

Proof. Let $r = \max_{1\le j\le l} r_j < 1$ and

$$a_1 = \left\| \sum_{j=1}^{l} p_j \circ w_j \right\|, \qquad a_2 = \left\| \sum_{j=l+1}^{m} p_j \circ w_j \right\|.$$

Since $a_2 < \varrho$, we can find $t_0 > 0$ such that

$$q_1 = \sup_{0\le t\le t_0} (a_2/\varrho)^{1-t}(a_1/\varrho)^t < 1.$$

Take $q_2 > 1$ such that $q_3 = q_1 q_2 < 1$. By Lemma 4.6, we can choose t_0 so small that $\#\mathcal{J}_n(t) < q_2^n$ for $0 < t \le t_0$ and $n > 0$. We claim that for

$J' = (j'_1, \ldots, j'_n) \in \mathcal{J}_n(t)$,

$$a_{J'} := a_{j'_1} \ldots a_{j'_n} \leq (q_1 \varrho)^n.$$

Indeed, if $a_1 \leq a_2$, then $a_{J'} \leq a_2^n = (a_2/\varrho)^n \varrho^n \leq (q_1\varrho)^n$ trivially. If $a_2 < a_1$, then by the definition of $\mathcal{J}_n(t)$, we have

$$a_{J'} \leq a_2^{n(1-t)} a_1^{nt} = ((a_2/\varrho)^{1-t}(a_1/\varrho)^t)^n \varrho^n \leq (q_1\varrho)^n.$$

It follows that

$$\sum_{J' \in \mathcal{J}_n(t)} a_{J'} \leq \#\mathcal{J}_n(t) \max_{J' \in \mathcal{J}_n(t)} a_{J'} \leq q_2^n (q_1 \varrho)^n \leq q_3^n \varrho^n.$$

Now for $J = (j_1, \ldots, j_n) \in \{1, \ldots, m\}^n$, let

$$j' = \begin{cases} 1 & \text{if } j \in \{1, \ldots, l\}, \\ 2 & \text{if } j \in \{l+1, \ldots, m\}, \end{cases}$$

and $\phi(J) = (j'_1, \ldots, j'_n)$. Then for any $I = (i_1, \ldots, i_n)$, $i_j = 1, 2$, we have

$$\begin{aligned} \sum_{J:\, \phi(J) = I} p_{w_J}(x) &= \sum_{j'_1 = i_1} \sum_{\phi(J) = (i_2, \ldots, i_n)} p_{j_1}(w_{j_1} w_J x) p_{w_J}(x) \\ &= \sum_{J:\, \phi(J) = (i_2, \ldots, i_n)} p_{w_J}(x) \sum_{j'_1 = i_1} p_{j_1}(w_{j_1} w_J x) \\ &\leq a_{i_1} \sum_{J:\, \phi(J) = (i_2, \ldots, i_n)} p_{w_J}(x) \leq \ldots \leq a_I. \end{aligned}$$

Therefore

$$\sum_{J:\, \phi(J) \in \mathcal{J}_n(t)} p_{w_J}(x) \leq \sum_{J' \in \mathcal{J}_n(t)} a_{J'} \leq q_3^n \varrho^n,$$

so that for n sufficiently large,

$$\begin{aligned} \sum_{|J| = n} p_{w_J}(x) r_J &\leq \sum_{J:\, \phi(J) \in \mathcal{J}_n(t)} p_{w_J}(x) + \sum_{J:\, \phi(J) \in \mathcal{J}'_n(t)} p_{w_J}(x) r^{nt} \\ &\leq q_3^n \varrho^n + r^{nt} \Big\| \sum_{|J| = n} p_{w_J}(\cdot) \Big\| < \varrho^n \end{aligned}$$

where $\mathcal{J}'_n(t) = \{J = (j_1, \ldots, j_n) : j_i = 1, 2,\ \#\{j_i = 1\} > nt\}$ is the complement of $\mathcal{J}_n(t)$. This completes the proof of the theorem. ∎

COROLLARY 4.8. *Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be a nonexpansive Dini system. If $w_1(x_1) = x_1$ and $p_1(x_1) = \max_{x \in K} p_1(w_1 x)$ for some $x_1 \in K$, and w_j is contractive for each $2 \leq j \leq m$, then T has the PF-property if and only if $p_1(x_1) < \varrho$.*

Proof. The sufficiency follows from Theorems 4.7 and 4.4. For the necessity we observe that there exists $0 < h \in C(K)$ such that $Th(x) = \varrho h(x)$.

Then by the continuity and positivity of h, we have

$$\varrho h(x_1) = p_1(x_1)h(x_1) + \sum_{j=2}^{m} p_j(w_j x_1)h(w_j x_1) > p_1(x_1)h(x_1).$$

Hence $p_1(x_1) < \varrho$. ∎

We can easily construct examples satisfying the assumptions of the above corollary, e.g.:

COROLLARY 4.9. *Let* $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ *be as in Corollary* 4.8. *If*

$$p_1(x_1) < \min_{x \in K} \sum_{j=1}^{m} p_j(w_j x),$$

then T *has the PF-property.*

We return to the example given at the end of Section 2. In that case

$$w_1(x) = \frac{x}{1+x}, \qquad w_2(x) = \frac{1}{2} + \frac{x}{2},$$

$$p_1(w_1 x) = w_1'(x) = \frac{1}{(1+x)^2}, \quad p_2(w_2 x) = w_2'(x) = \frac{1}{2},$$

and

$$Tf(x) = \frac{1}{(1+x)^2} f\left(\frac{x}{1+x}\right) + \frac{1}{2} f\left(\frac{1}{2} + \frac{x}{2}\right).$$

We show that $\varrho = 1$. First we observe that

$$T^n 1(0) = \sum_{|J|=n} p_{w_J}(0) \geq 1,$$

hence $\varrho = \lim_{n\to\infty} \|T^n 1\|^{1/n} \geq 1$. If $\varrho > 1$, then by Corollary 4.8, there exists a ϱ-eigenfunction h. By integrating $\varrho h(x) = Th(x)$ over $[0,1]$, we see that $\varrho C = C$ where C is the integral of h. This contradicts $\varrho > 1$ and hence $\varrho \leq 1$. This implies $\varrho = 1$.

We have seen in Section 2 that $\varrho = 1$ has no eigenfunction; this is also clear from Corollary 4.8. If we redefine the operator T as

$$Tf(x) = \frac{1}{(1+x)^2} f\left(\frac{x}{1+x}\right) + \lambda f\left(\frac{1}{2} + \frac{x}{2}\right),$$

then for $\lambda > 3/4$, Corollary 4.9 implies that a ϱ-eigenfunction exists.

5. Self-conformal maps. We assume the interior X° of X is nonempty and $\overline{X^\circ} = X$. We say that a map $w : X \to X$ is *self-conformal* if w is continuously differentiable on a neighborhood of X and $|w'(x)|$ is a self-similar matrix ($|w'(x)|$ denotes a matrix norm). In this section we will consider one-to-one, self-conformal weakly contractive maps with

$$0 < \inf_{x,j} |w_j'(x)| \le \sup_{x,j} |w_j'(x)| \le 1.$$

The IFS $\{w_j\}_{j=1}^m$ is said to satisfy the *open set condition* (OSC) if there exists a bounded open set U in X such that

$$w_j(U) \subset U \quad \text{and} \quad w_i(U) \cap w_j(U) \neq \emptyset, \quad i \neq j.$$

It is easy to see that if $\{w_j\}_{j=1}^m$ are weakly contractive and satisfy the OSC, then $K \subseteq \overline{U}$, $K = \bigcap_{n=1}^\infty \bigcup_{|J|=n} w_J(\overline{U})$ and $\lim_{|J|\to\infty} w_J(\overline{U}) = 0$.

THEOREM 5.1. *Suppose $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ is a weakly contractive self-conformal Dini system which satisfies the OSC. If $w_1, \ldots, w_l$ are contractive for some $0 \le l \le m$ and*

$$\max_{\substack{l+1\le j\le m \\ x\in K}} p_j(w_j x) < \varrho, \tag{5.1}$$

then T has the PF-property.

Note that this improves the condition $\max_{x\in K} \sum_{j=l+1}^m p_j(w_j x) < \varrho$ of Theorem 4.7. It follows directly from (4.10) that (5.1) is satisfied if

$$\max_{\substack{l+1\le j\le m \\ x\in K}} p_j(w_j x) < \min_{x\in K} \sum_{j=1}^m p_j(w_j x).$$

Proof. We divide the proof into two cases: (i) $l = 0$ (i.e., none of the maps are contractive) and (ii) $l \ge 1$.

(i) $l = 0$: Let $d_n := \max_{|J|=n} |w_J(X)|$ and $\delta(t) = \max_{1\le j\le m} \alpha_{w_j'}(t)$. Since $|w_j'(x)|$ is continuous and positive on the compact set X, there exists $c_1 > 0$ such that

$$\frac{|w_j'(x)|}{|w_j'(y)|} \le 1 + c_1\delta(|x-y|) \qquad \forall x, y \in X.$$

Hence for $J = (j_1, \ldots, j_n)$,

$$\begin{aligned}\frac{|w_J'(x)|}{|w_J'(y)|} &= \prod_{k=1}^n \frac{|w_{j_k}'(w_{j_{k+1}} \circ \ldots \circ w_{j_n} x)|}{|w_{j_k}'(w_{j_{k+1}} \circ \ldots \circ w_{j_n} y)|} \\ &\le \prod_{k=1}^n (1 + c_1\delta(|w_{J|_k^n}(K)|)) \le \prod_{k=1}^n (1 + c_1\delta(d_{n-k})).\end{aligned}$$

Let

$$c = \max_{\substack{l+1\le j\le m \\ x\in K}} p_j(w_j x).$$

Choose $\theta, \varepsilon > 0$ such that $c\varrho^{-1} < \theta < 1$ and $e^\varepsilon \theta < 1$. Since $\lim_n d_n = 0$, $\delta(t)$ is continuous and $\delta(0) = 0$, there exists $k_0 > 0$ such that $\delta(d_k) < c_1{}^{-1}\varepsilon$

whenever $k \geq k_0$. Hence there exists $a > 0$ such that for any integer $n > 0$,

$$\sup_{x \neq y} \frac{|w'_J(x)|}{|w'_J(y)|} \leq a(1+\varepsilon)^n \leq a e^{n\varepsilon}.$$

Let $\gamma_J = \inf_{x\in X} |w'_J(x)|$ and $R_J = \sup_{x\in X} |w'_J(x)|$. Then $R_J \leq a e^{n\varepsilon}\gamma_J$. Let

$$\Omega(n) = \{J : |J| = n,\ \gamma_J \leq \theta^n\}, \qquad \Omega'(n) = \{J : |J| = n,\ \gamma_J > \theta^n\}.$$

Let U be the open set in the OSC, and let B be a ball in U. Then for any $J \in \Omega'(n)$,

$$|w_J(x) - w_J(y)| \geq r_J|x-y| \geq \theta^n|x-y| \qquad \forall x, y \in B.$$

It follows that $w_J(U)$ contains a ball of radius $c_2\theta^n$ for some $c_2 > 0$ independent of J. This together with the disjointness of the $w_J(U)$ (by OSC) implies that $\#\Omega'(n) < c_3\theta^{-n}$ for some $c_3 > 0$. On the other hand $R_J \leq a(e^\varepsilon\theta)^n$ for any $J \in \Omega(n)$. Therefore

$$\begin{aligned}(5.2)\qquad &\lim_n \Big(\max_{x\in K} \sum_{|J|=n} p_{w_J}(x)R_J\Big)^{1/n} \\ &\leq \lim_n \Big(\max_{x\in K} \sum_{J\in\Omega(n)} p_{w_J}(x)R_J + \max_{x\in K} \sum_{J\in\Omega'(n)} p_{w_J}(x)R_J\Big)^{1/n} \\ &\leq \lim_n \left(\|T^n 1\| a(e^\varepsilon\theta)^n + c_3 c^n \theta^{-n}\right)^{1/n} < \varrho.\end{aligned}$$

Theorem 4.4 applies and T has the PF-property.

(ii) $l \geq 1$: The assertion is proved in [FL] if all the maps are contractive, hence we assume that $1 \leq l \leq m-1$. Let c be defined as above and let

$$b = \max_{\substack{1\leq j\leq l \\ x\in K}} p_j(w_j x), \qquad R = \max_{\substack{1\leq j\leq l \\ x\in K}} |w'_j(x)| < 1.$$

If $b \leq c$, then

$$\max_{\substack{1\leq j\leq m \\ x\in K}} p_j(w_j x) = \max_{\substack{l+1\leq j\leq m \\ x\in K}} p_j(w_j x) < \varrho,$$

and the proof of (i) applies. Hence we assume $b > c$. We choose θ such that

$$c\varrho^{-1} < \theta < 1 \quad \text{and} \quad \theta^{-1}(bc^{-1})^{\log\theta/\log R} < \varrho c^{-1}.$$

Let $\Omega(n)$ and $\Omega'(n)$ be defined as above. For any $J \in \Omega'(n)$, set

$$k_J = \#\{j_i : J = (j_1, \ldots, j_i, \ldots, j_n) \in \Omega'(n),\ 1 \leq j_i \leq l\}.$$

Then $p_{w_J}(x) \leq c^{n-k_J} b^{k_J}$ and $\#\Omega'(n) \leq c_3\theta^{-n}$ as in (i) and

$$\begin{aligned}(5.3)\qquad \sum_{J\in\Omega'(n)} p_{w_J}(x)R_J &\leq c_3\theta^{-n} \max_{J\in\Omega'(n)} c^{n-k_J} b^{k_J} \\ &= c_3(c\theta^{-1})^n \max_{J\in\Omega'(n)} (bc^{-1})^{k_J}.\end{aligned}$$

Let $k_n = \max\{k_J : J \in \Omega'(n)\}$. Since $\theta^n < \gamma_J \le R^{k_J}$ for any $J \in \Omega'(n)$, this implies that $\theta^n \le R^{k_n}$ and

$$(\max_{J\in\Omega'(n)} (bc^{-1})^{k_J})^{1/n} \le (bc^{-1})^{k_n/n} \le (bc^{-1})^{\log\theta/\log R} < \theta\varrho c^{-1}.$$

Hence by (5.3),

$$\overline{\lim_n}\Big(\max_{x\in K} \sum_{J\in\Omega'(n)} p_{w_J}(x)R_J\Big)^{1/n} < (c\theta^{-1})(\theta\varrho c^{-1}) = \varrho.$$

The argument in (5.2) implies that $\lim_n(\max_{x\in K}\sum_{|J|=n} p_{w_J}(x)R_J)^{1/n} < \varrho$ and the proof is complete. ∎

In the above proof we need to use the weak contractivity of the w_j's ($d_n := \max_{|J|=n}|w_J(X)| \to 0$). We do not know if we can replace such maps by nonexpansive maps. Concerning the OSC, Schief [S] proved that for self-similar contractive maps, the OSC implies the strong OSC (SOSC), i.e., the bounded open set U in the definition intersects K. Recently Peres *et al.* [PRS] proved that the statement can be extended to self-conformal contractive maps. Lau *et al.* [LRY] gave another simple proof. The SOSC is technically important and it plays an important role in the study of the Hausdorff dimension and Hausdorff measure of the invariant set (see [Fal] and [FL]); moreover, it implies that $\mu(K_I \cap K_J) = 0$, $I \ne J$, $|I| = |J|$, for any self-conformal measure [FL]. We conjecture the same also holds for weakly contractive self-conformal maps.

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