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ON THE BOUNDARY OF ATTRACTORS WITH NON-VOID INTERIOR

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ABSTRACT. Let $\{f_i\}_{i=1}^N$ be a family of N contractive mappings on $\mathbb{R}^d$ such that the attractor K has nonvoid interior. We show that if the f_i's are injective, have non-vanishing Jacobian on K, and $f_i(K) \cap f_j(K)$ have zero Lebesgue measure for $i \neq j$, then the boundary ∂K of K has measure zero. In addition if the f_i's are affine maps, then the conclusion can be strengthened to $\dim_H(\partial K) < d$. These improve a result of Lagarias and Wang on self-affine tiles.

1. INTRODUCTION

A function $f : \mathbb{R}^d \to \mathbb{R}^d$ is called a *contraction* if $\|f(\mathbf{x}) - f(\mathbf{y})\| \leq r\|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, where $r < 1$ is a constant. If equality holds, then f is called a *similarity* and r is called the *contractive ratio* of f. Let $\{f_i\}_{i=1}^N$ be a family of contractions on $\mathbb{R}^d$ and let K be the corresponding attractor. The Hausdorff dimension and the α-dimensional Hausdorff measure of K are denoted by $\dim_H(K)$ and $\mathcal{H}^\alpha(K)$ respectively. We say that $\{f_i\}_{i=1}^N$ satisfies the *open set condition* (OSC) if there exists an open set O such that $\bigcup_{i=1}^N f_i(O) \subset O$ and $f_i(O) \cap f_j(O) = \emptyset$ for $i \neq j$. It is well-known that if the contractions f_i are all similarities, then OSC implies that the Hausdorff dimension of K equals the *similarity dimension* α which is the unique number determined by $\sum_{i=1}^N r_i^\alpha = 1$. The work of Bandt and Graf [2] and Schief [12] showed that OSC is equivalent to $\mathcal{H}^\alpha(K) > 0$. If $\alpha = d$, the condition is further equivalent to the interior $K^\circ \neq \emptyset$.

Let $\mathbb{M}_d(\mathbb{R})$ denote the class of real $d \times d$ matrices and let μ denote the Lebesgue measure on $\mathbb{R}^d$. A matrix B is called *expanding* if all of its eigenvalues satisfy $|\lambda_i| > 1$. Let $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_N\} \subset \mathbb{R}^d$ and let $f_i(\mathbf{x}) = B^{-1}\mathbf{x} + \mathbf{a}_i$, $\mathbf{x} \in \mathbb{R}^d$, be the affine transformations. Then under an appropriate metric on $\mathbb{R}^d$, the f_i's are all contractions [10]. It follows that the attractor K exists [8]. If in addition $|\det B| = N$, then $\mu(K) > 0$ is equivalent to $K^\circ \neq \emptyset$. In this case K is called a *self-affine tile*. Lagarias and Wang [10] proved that the boundary ∂K of such K

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has Lebesgue measure zero. In this note we will consider a few extensions of this result. We show that for a very general class of self-affine attractors in $\mathbb{R}^d$, the Hausdorff dimension of the boundary is strictly less than d.

For a function $f : \mathbb{R}^d \to \mathbb{R}^d$ and a compact set $E \subset \mathbb{R}^d$, we say that $f \in C^1(E)$ if f has continuous first order partial derivatives on a neighborhood of E. The Jacobi determinant of f at $\mathbf{x}$ is denoted by $J_f(\mathbf{x})$.

Theorem 1.1. *Let $\{f_i\}_{i=1}^N$ be a family of contractions on $\mathbb{R}^d$ and K the corresponding attractor with non-void interior. Suppose $f_i \in C^1(K)$, $1 \le i \le N$, are injective on K with $J_{f_i}(\mathbf{x}) \neq 0$, and $\mu(f_i(K) \cap f_j(K)) = 0$ when $i \neq j$. Then $\mu(\partial K) = 0$.*

It is easy to verify that the condition $\mu(f_i(K) \cap f_j(K)) = 0$ is equivalent to

$$\sum_{i=1}^N \mu(f_i(K)) = \mu(K), \text{ or } \int_K \sum_{i=1}^N |J_{f_i}(\mathbf{x})| \, d\mathbf{x} = \mu(K).$$

One sufficient condition for the above equalities to hold is $\sum_{i=1}^N |J_{f_i}(\mathbf{x})| = 1$ on all points of K. In particular when $f_i(\mathbf{x}) = A_i\mathbf{x} + \mathbf{a}_i$, $A_i \in \mathbb{M}_d(\mathbb{R})$, are affine contractions, then the condition reduces to $\sum_{i=1}^N |\det A_i| = 1$. In this case, we prove the following stronger result.

Theorem 1.2. *Let $f_i(\mathbf{x}) = A_i\mathbf{x} + \mathbf{a}_i$, $1 \le i \le N$, be a family of affine contractions on $\mathbb{R}^d$ and let K be the corresponding attractor. If $K^\circ \neq \emptyset$ and $\sum_{i=1}^N |\det A_i| = 1$, then $\dim_H(\partial K) < d$.*

Note that in the theorem we allow some A_i's to be singular. Using this theorem, we prove the following corollaries.

Corollary 1.3. *Let $\{f_i\}_{i=1}^N$ be a family of contractive similarities on $\mathbb{R}^d$ and K the corresponding attractor. If $K^\circ \neq \emptyset$ and the similarity dimension of $\{f_i\}_{i=1}^N$ is d, then $\dim_H(\partial K) < d$.*

Corollary 1.4. *Let $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_N\} \subset \mathbb{R}^d$ and let $B \in \mathbb{M}_d(\mathbb{R})$ be expanding with $|\det B| = N$. If the attractor K of $\left\{f_i : f_i(\mathbf{x}) = B^{-1}\mathbf{x} + \mathbf{a}_i\right\}_{i=1}^N$ has non-void interior, then $\dim_H(\partial K) < d$.*

The second corollary improves Lagarias and Wang's result on the boundary of self-affine tiles. As mentioned before, the condition in the above two corollaries that K has non-void interior is equivalent to the condition that K has positive Lebesgue measure. We point out that it was proved recently [9] that the Hausdorff dimension of the boundary of a self-affine tile in $\mathbb{R}^d$ can be arbitrarily close to d.

Note that in [1] Bandt also considered the rotations and reflections of a tile. We call a finite group $\mathbb{W}$ of matrices with determinant ± 1 a *symmetry group* of an expanding matrix B if $B\mathbb{W} = \mathbb{W}B$. Let $w_i \in \mathbb{W}$, $\mathbf{a}_i \in \mathbb{R}^d$ and $f_i(\mathbf{x}) = w_i B^{-1}\mathbf{x} + \mathbf{a}_i$, $i = 1, \cdots, N$. Then $\{f_i\}_{i=1}^N$ can generate more exotic tiles such as Levy's curve, Heighway dragon, etc. [1]. Theorem 1.2 also applies to these tiles.

Corollary 1.5. *Let $B \in \mathbb{M}_d(\mathbb{R})$ be expanding with $|\det B| = N$ and $\mathbb{W}$ be a symmetry group of B. Let $f_i(\mathbf{x}) = w_i B^{-1}\mathbf{x} + \mathbf{a}_i$, $w_i \in \mathbb{W}$, $\mathbf{a}_i \in \mathbb{R}^d$, $i = 1, \cdots, N$. If the attractor K of $\{f_i\}_{i=1}^N$ has non-void interior, then $\dim_H(\partial K) < d$.*

We remark that if K is the attractor generated by affine transformations consisting of one single matrix, then there are simple criteria to determine that K has non-void interior [10]. However very little is known if there is more than one matrix (see [1] for the case in Corollary 1.5). The estimation of the dimension of the boundary is far from being understood. It seems that only for a few particular classes of self-similar tiles ([7], [4] and [13]) has the exact dimension been calculated.

For an attractor K of similarities $\{f_i\}_{i=1}^N$ in $\mathbb{R}^d$ with similarity dimension α, it is well-known that if $\{f_i\}_{i=1}^N$ satisfies OSC, then $\mathcal{H}^\alpha(f_i(K) \cap f_j(K)) = 0$ for $i \neq j$ [5, Corollary 8.7]. Our theorem sharpens this result.

Theorem 1.6. *Let $\{f_i\}_{i=1}^N$ be a family of contractive similarities on $\mathbb{R}^d$ with similarity dimension α, $0 < \alpha \leq d$, and let K be the corresponding self-similar set. If $\{f_i\}_{i=1}^N$ satisfies the OSC, then $\dim_H(f_i(K) \cap f_j(K)) < \alpha$ for $i \neq j$.*

2. Definitions and preliminaries

Let $\mathbb{N}$ be the set of natural numbers. For $\mathbb{S} = \{1, 2, \cdots, N\}$, let $\mathbb{S}^n = \underbrace{\mathbb{S} \times \cdots \times \mathbb{S}}_{n}$ and $\mathbb{S}^* = \bigcup_{n\in\mathbb{N}} \mathbb{S}^n$. The length of $\mathbf{s} = (s_1 \cdots s_n) \in \mathbb{S}^n$ is denoted by $|\mathbf{s}| (= n)$. If $\mathbf{i} = (i_1 i_2 \cdots i_{n_1})$, $\mathbf{j} = (j_1 j_2 ... j_{n_2})$, then we define

$$\mathbf{ij} = (i_1 i_2 \cdots i_{n_1} j_1 j_2 ... j_{n_2}).$$

For a subset $E \subset \mathbb{R}^d$, its diameter is defined as $|E| = \sup\{|\mathbf{x} - \mathbf{y}| : \mathbf{x}, \mathbf{y} \in E\}$. For $f_i : \mathbb{R}^d \to \mathbb{R}^d$, $i \in \mathbb{S}$, we define $E_{\mathbf{s}} = f_{\mathbf{s}}(E) = f_{s_1} \circ f_{s_2} \circ \cdots \circ f_{s_n}(E)$. It is easy to see that if all f_i's are contractive, then $f_{\mathbf{s}}$ is also contractive and its contractive ratio is $r_{\mathbf{s}} = r_{s_1} r_{s_2} \cdots r_{s_n}$.

We use $B_a(\mathbf{x})$ to denote the closed ball with center $\mathbf{x}$ and radius a. Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^d$ and the norm of a matrix $A \in \mathbb{M}_d(\mathbb{R})$ is

$$\|A\| = \max\left\{\frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} : \mathbf{x} \in \mathbb{R}^d, \|\mathbf{x}\| \neq 0\right\}.$$

The spectral radius of A is $\lambda_{\max} = \max_{1\leq i\leq d} |\lambda_i|$, where λ_i are the eigenvalues of A. If Q is the closed unit ball in $\mathbb{R}^d$ and $f(\mathbf{x}) = A\mathbf{x} + \mathbf{a}$ is an affine mapping with A non-singular, then $A(Q)$ is an ellipsoid. The lengths of the principle semi-axes of $A(Q)$ are singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_d > 0$ of A. These singular values are also the positive roots of the eigenvalues of $A^T A$, where A^T is the transpose of A. The norm and the singular values of A have the following relationships:

$$\sigma_1 = \max\left\{\frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} : \|\mathbf{x}\| = 1\right\} = \|A\|,$$

$$\sigma_d = \min\left\{\frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} : \|\mathbf{x}\| \neq 0\right\} = \min\left\{\frac{\|\mathbf{y}\|}{\|A^{-1}\mathbf{y}\|} : \|\mathbf{y}\| \neq 0\right\} = 1/\left\|A^{-1}\right\|.$$

Also we have

$$|\det A| = \sqrt{\det(A^T A)} = \sigma_1 \sigma_2 \cdots \sigma_d.$$

Using the spectral radius formula $\lambda_{\max} = \lim_{n\to\infty} \|A^n\|^{1/n}$ [11, Theorem 10.13], it is easy to see the following:

Lemma 2.1. *Let $A \in \mathbb{M}_d(\mathbb{R})$ with spectral radius $\lambda_{\max}$ and $\rho > \lambda_{\max}$. Then there exists a constant $c > 0$ depending on A such that for any $n \in \mathbb{N}$, $\|A^n\| \le c \cdot \rho^n$.*

3. Proofs of the results

Proof of Theorem 1.1. We know that for any $n \in \mathbb{N}$,

$$K = \bigcup_{i=1}^{N} f_i(K) = \bigcup_{\mathbf{s}\in\mathbb{S}^n} f_{\mathbf{s}}(K) = \bigcup_{\mathbf{s}\in\mathbb{S}^n} K_{\mathbf{s}}.$$

Since f_i are contractive, $\max_{\mathbf{s}\in\mathbb{S}^n} |K_{\mathbf{s}}| \to 0$ when $n \to \infty$. We can find $\varepsilon > 0$, $\mathbf{x}_0 \in K$ and $\mathbf{k} \in \mathbb{S}^m$ for some large m such that

$$K_{\mathbf{k}} \subset B_\varepsilon(\mathbf{x}_0) \subset K^\circ \neq \emptyset.$$

Since $J_{f_i}(\mathbf{x}) \neq 0$ and f_i are injective for all $i = 1, \cdots, N$, it is easy to see that for any $\mathbf{s} \in \mathbb{S}^n$, $f_{\mathbf{s}}$ is a homeomorphism between K and $f_{\mathbf{s}}(K)$. It follows that $f_{\mathbf{s}} \in C^1(K)$, $f_{\mathbf{s}}^{-1} \in C^1(K_{\mathbf{s}})$, $K_{\mathbf{s}}^\circ = f_{\mathbf{s}}(K^\circ)$ and $\partial K_{\mathbf{s}} = f_{\mathbf{s}}(\partial K)$. We claim that if $\mathbf{i}, \mathbf{j} \in \mathbb{S}^m$ and $\mathbf{i} \neq \mathbf{j}$, then $\mu(K_{\mathbf{i}} \cap K_{\mathbf{j}}) = 0$. Indeed, suppose $\mathbf{i} = (i_1 \cdots i_m)$ and $\mathbf{j} = (j_1 \cdots j_m)$. Let l be the smallest integer such that $i_l \neq j_l$. Then

$$\begin{aligned} \mu(K_{\mathbf{i}} \cap K_{\mathbf{j}}) &= \mu\left(f_{i_1\cdots i_{l-1}}(K_{i_l\cdots i_m}) \cap f_{i_1\cdots i_{l-1}}(K_{j_l\cdots j_m})\right) \\ &= \mu\left(f_{i_1\cdots i_{l-1}}(K_{i_l\cdots i_m} \cap K_{j_l\cdots j_m})\right) \le \mu\left(f_{i_1\cdots i_{l-1}}(K_{i_l} \cap K_{j_l})\right) \\ &= \int_{K_{i_l}\cap K_{j_l}} \left|J_{f_{i_1\cdots i_{l-1}}}(\mathbf{x})\right| d\mathbf{x} = 0, \end{aligned}$$

where the second equality holds because $f_{i_1\cdots i_{l-1}}$ is injective, and the last equality holds because $\mu(K_{i_l} \cap K_{j_l}) = 0$. This proves the claim.

Now since $\partial K_{\mathbf{k}} \subset K_{\mathbf{k}} \subset K^\circ$, for $\mathbf{x} \in \partial K_{\mathbf{k}}$, we can find a sequence $\{\mathbf{y}_i\}_{i=1}^\infty$ such that $\mathbf{y}_i \to \mathbf{x}$ and $\mathbf{y}_i \in K^\circ \backslash K_{\mathbf{k}}$. It follows that $\mathbf{y}_i \in K_{\mathbf{t}_i} \backslash K_{\mathbf{k}}$ for some $\mathbf{t}_i \in \mathbb{S}^m$. Since there are only finitely many elements in $\mathbb{S}^m$, we can assume, by passing to subsequence, that $\mathbf{y}_i \in K_{\mathbf{t}}$ for a fixed $\mathbf{t} \in \mathbb{S}^m$. Since $\mathbf{x}$ is the limit point of the sequence and $K_{\mathbf{t}}$ is compact, we have $\mathbf{x} \in K_{\mathbf{t}}$. This is true for any $\mathbf{x} \in \partial K_{\mathbf{k}}$. Therefore $\partial K_{\mathbf{k}} \subset K_{\mathbf{k}} \cap \left(\bigcup_{\mathbf{t}\in\mathbb{S}^m, \mathbf{t}\neq\mathbf{k}} K_{\mathbf{t}}\right)$ and hence

$$\mu(\partial K_{\mathbf{k}}) \le \mu\left(K_{\mathbf{k}} \cap \left(\bigcup_{\mathbf{t}\in\mathbb{S}^m, \mathbf{t}\neq\mathbf{k}} K_{\mathbf{t}}\right)\right) \le \sum_{\mathbf{t}\in\mathbb{S}^m, \mathbf{t}\neq\mathbf{k}} \mu(K_{\mathbf{k}} \cap K_{\mathbf{t}}) = 0$$

by the claim. Note that $\partial K = f_{\mathbf{k}}^{-1}(\partial K_{\mathbf{k}})$; we have

$$\mu(\partial K) = \int_{\partial K_{\mathbf{k}}} \left|J_{f_{\mathbf{k}}^{-1}}(\mathbf{x})\right| d\mathbf{x} = 0.$$

□

Proof of Theorem 1.2. For any $n \in \mathbb{N}$, we have

$$K = \bigcup_{i=1}^{N} f_i(K) = \bigcup_{\mathbf{s}\in\mathbb{S}^n} f_{\mathbf{s}}(K) = \bigcup_{\mathbf{s}\in\mathbb{S}^n} K_{\mathbf{s}},$$

where

$$f_{\mathbf{s}}(\mathbf{x}) = A_{s_1}\cdots A_{s_n}\mathbf{x} + A_{s_1}\cdots A_{s_{n-1}}\mathbf{a}_{s_n} + \cdots + A_{s_1}\mathbf{a}_{s_2} + \mathbf{a}_{s_1}.$$

We use $A_{\mathbf{s}}$ to denote $A_{s_1}\cdots A_{s_n}$. Since f_i's are all contractions, $\|A_i\| < 1$ for $i = 1, 2, \cdots, N$. Let $\rho = \max_{1\le i\le N}\{\|A_i\|\}$, $\kappa = |K|$. Then $\rho < 1$ and $|K_{\mathbf{s}}| \le \|A_{\mathbf{s}}\|\,\kappa \le \rho^n\kappa$. Since $K^\circ \ne \emptyset$, there is an open ball $B_\varepsilon(\mathbf{x}_0) \subset K^\circ$. Pick an integer m big enough so that $\rho^m\kappa < \frac{\varepsilon}{2}$. We have $B_{\varepsilon/2}(\mathbf{x}_0) \subset \bigcup_{\mathbf{s}\in\mathbb{S}^m} K_{\mathbf{s}}$. If $A_{\mathbf{s}}$ is singular, then $K_{\mathbf{s}}$ has volume zero. So there exists a $\mathbf{k} \in\mathbb{S}^m$ such that $A_{\mathbf{k}}$ is non-singular and $K_{\mathbf{k}} \cap B_{\varepsilon/2}(\mathbf{x}_0) \ne \emptyset$. It follows that $K_{\mathbf{k}} \subset B_\varepsilon(\mathbf{x}_0) \subset K^\circ$.

Now without loss of generality, we suppose $A_1, \cdots, A_M$, $0 \le M < N$, are singular matrices and $A_{M+1}, \cdots, A_N$ are non-singular. Set

$$\widetilde{\mathbb{S}^*} = \{\mathbf{s} : \mathbf{s} \in\mathbb{S}^*, s_i \notin \{1, 2, \cdots, M\}, i = 1, \cdots, |\mathbf{s}|\}.$$

Then for any $\mathbf{s} \in\widetilde{\mathbb{S}^*}$, $A_{\mathbf{s}}$ is non-singular, and for any $\mathbf{s} \in\mathbb{S}^*\backslash\widetilde{\mathbb{S}^*}$, $A_{\mathbf{s}}$ is singular. Let $\mathbf{j} \in\widetilde{\mathbb{S}^*}$. Then

$$K_{\mathbf{jk}} = f_{\mathbf{jk}}(K) = f_{\mathbf{j}}(K_{\mathbf{k}}) \subset f_{\mathbf{j}}(K^\circ) = (f_{\mathbf{j}}(K))^\circ \subset K^\circ.$$

Let $E = K_{\mathbf{k}} \cup \left(\bigcup_{\mathbf{j}\in\widetilde{\mathbb{S}^*}} K_{\mathbf{jk}}\right)$ and $F = K\backslash E$. Then $E \subset K^\circ$ and

$$\partial K = K\backslash K^\circ \subset K\backslash E = F.$$

For $l \in \mathbb{N}$ and $\mathbf{s} \in\mathbb{S}^{ml}$, we write $\mathbf{s} = (\mathbf{s}_1, \cdots, \mathbf{s}_l) \in \mathbb{S}^m \times \cdots \times \mathbb{S}^m = \mathbb{S}^{ml}$. Let

$$J = \left\{\mathbf{s} : \mathbf{s} \in \left(\mathbb{S}^{ml} \cap \widetilde{\mathbb{S}^*}\right), \mathbf{s}_j = \mathbf{k} \text{ for some } j\right\} \text{ and } L = \mathbb{S}^{ml}\backslash J.$$

Then $K = \left(\bigcup_{\mathbf{s}\in J} K_{\mathbf{s}}\right) \cup \left(\bigcup_{\mathbf{s}\in L} K_{\mathbf{s}}\right)$ and

$$F = K\backslash E \subset K\backslash\left(\bigcup_{\mathbf{s}\in J} K_{\mathbf{s}}\right) \subset \bigcup_{\mathbf{s}\in L} K_{\mathbf{s}}.$$

If we let U be a ball of radius $a > 0$ and contain K, then

$$\partial K \subset F \subset \bigcup_{\mathbf{s}\in L} K_{\mathbf{s}} \subset \bigcup_{\mathbf{s}\in L} U_{\mathbf{s}}.$$

From $\sum_{i=1}^{N} |\det A_i| = 1$, it is easy to show that $\sum_{\mathbf{s}\in L} |\det A_{\mathbf{s}}| = (1 - |\det A_{\mathbf{k}}|)^l$. We will use the identity later.

For each non-singular $A_{\mathbf{s}}$, we use an idea from Falconer [6, p132] to get an estimate. We know that $U_{\mathbf{s}}$ is an ellipsoid with principal semi-axes $a\sigma_1(\mathbf{s}) \ge a\sigma_2(\mathbf{s}) \ge \cdots \ge a\sigma_d(\mathbf{s}) > 0$, where $\sigma_i(\mathbf{s})\,(1 \le i \le d)$ are the singular values of $A_{\mathbf{s}}$. The ellipsoid is contained in a rectangular parallelepiped P of side lengths $2a\sigma_1(\mathbf{s})$, $2a\sigma_2(\mathbf{s})$,

$\cdots$, $2a\sigma_d(\mathbf{s})$. We may cover P by at most γ cubes of side $2a\sigma_d(\mathbf{s})$, where

$$\gamma = \prod_{i=1}^{d-1} \frac{4a\sigma_i(\mathbf{s})}{2a\sigma_d(\mathbf{s})} = 2^{d-1}\left(\prod_{i=1}^{d}\sigma_i(\mathbf{s})\right)(\sigma_d(\mathbf{s}))^{-d} = 2^{d-1}|\det A_{\mathbf{s}}|(\sigma_d(\mathbf{s}))^{-d}.$$

The diameter of each cube is $2a\sigma_d(\mathbf{s})\sqrt{d}$. Note that $\sigma_d(\mathbf{s}) \le \sigma_1(\mathbf{s}) = \|A_{\mathbf{s}}\| \le \rho^{ml}$. Let $\delta = 2a\rho^{ml}\sqrt{d}$. Then

$$\mathcal{H}_\delta^\beta(U_{\mathbf{s}}) \le \gamma\left(2a\sigma_d(\mathbf{s})\sqrt{d}\right)^\beta = c_\beta|\det A_{\mathbf{s}}|(\sigma_d(\mathbf{s}))^{\beta-d},$$

where $c_\beta = 2^{d-1}\left(2a\sqrt{d}\right)^\beta$ is a constant depending on β.

For each singular matrix $A_{\mathbf{s}}, U_{\mathbf{s}}$ is contained in a hyperplane. So if $\beta > d-1$, then $\mathcal{H}^\beta(U_{\mathbf{s}}) = 0$ and hence $\mathcal{H}_\delta^\beta(U_{\mathbf{s}}) = 0$ for any $\delta > 0$. Let $\widetilde{L} = L \cap \widetilde{\mathbb{S}^*}$. Then for every $\mathbf{s} \in L\backslash\widetilde{L}$, $A_{\mathbf{s}}$ is singular, and hence for $\beta > d-1$,

$$\mathcal{H}_\delta^\beta(F) \le \mathcal{H}_\delta^\beta\left(\bigcup_{\mathbf{s}\in L} U_{\mathbf{s}}\right) = \mathcal{H}_\delta^\beta\left(\bigcup_{\mathbf{s}\in\widetilde{L}} U_{\mathbf{s}}\right) \le \sum_{\mathbf{s}\in\widetilde{L}} c_\beta|\det A_{\mathbf{s}}|(\sigma_d(\mathbf{s}))^{\beta-d}.$$

Since $\sigma_d(\mathbf{s})$ is the smallest singular value of $A_{\mathbf{s}}$,

$$\sigma_d(\mathbf{s}) = \frac{1}{\left\|(A_{\mathbf{s}})^{-1}\right\|} \ge \frac{1}{\left\|A_{s_{ml}}^{-1}\right\|\cdots\left\|A_{s_1}^{-1}\right\|} = \tau_{M+1}^{n_1}\cdots\tau_N^{n_{N-M}},$$

where $n_1 + n_2 + \cdots + n_{N-M} = ml$ and $\tau_i = 1/\left\|A_i^{-1}\right\|$ is the smallest singular value of A_i, $M+1 \le i \le N$. It is clear that $0 < \tau_i \le \|A_i\| < 1$. Let $\tau = \min\{\tau_i : i = M+1, \cdots, N\}$; then $\sigma_d(\mathbf{s}) \ge \tau^{ml}$. So for $d-1 < \beta < d$,

$$\begin{aligned}\mathcal{H}_\delta^\beta(F) &\le \sum_{\mathbf{s}\in\widetilde{L}} c_\beta|\det A_{\mathbf{s}}|\tau^{ml(\beta-d)} = c_\beta\tau^{ml(\beta-d)}\sum_{\mathbf{s}\in\widetilde{L}}|\det A_{\mathbf{s}}| \\ &= c_\beta\tau^{ml(\beta-d)}\sum_{\mathbf{s}\in L}|\det A_{\mathbf{s}}| = c_\beta\left(\tau^{m(\beta-d)}(1-|\det A_{\mathbf{k}}|)\right)^l.\end{aligned}$$

Let $\tau^{m(\beta-d)}(1-|\det A_{\mathbf{k}}|) = 1$, i.e.,

$$\beta = d - \frac{\ln(1-|\det A_{\mathbf{k}}|)}{m\ln\tau} < d.$$

Then $\mathcal{H}_\delta^\beta(F) < c_\beta$. Let $l \to \infty$; then $\delta \to 0$ and $\mathcal{H}^\beta(F) \le c_\beta$. It follows that $\dim_H(F) \le \beta < d$. Since $\partial K \subset F$, we conclude that $\dim_H(\partial K) < d$. $\square$

Proof of Corollary 1.3. Let $f_i(\mathbf{x}) = A_i\mathbf{x} + \mathbf{a}_i$ be the similarities and let r_i be the contractive ratios. Then $|\det A_i| = r_i^d$. It follows from the assumption that $\sum_{i=1}^{N}|\det A_i| = \sum_{i=1}^{N} r_i^d = 1$ and Theorem 1.2 applies. $\square$

Proof of Corollary 1.4. Let $A = B^{-1}$. Then $f_i(\mathbf{x}) = A\mathbf{x} + \mathbf{a}_i$, $i = 1, \cdots, N$. For any $n \in \mathbb{N}$, $K = \bigcup_{\mathbf{s}\in\mathbb{S}^n} f_{\mathbf{s}}(K)$ and

$$f_{\mathbf{s}}(\mathbf{x}) = A^n\mathbf{x} + A^{n-1}\mathbf{a}_{s_n} + \cdots + A\mathbf{a}_{s_2} + \mathbf{a}_{s_1}.$$

Suppose the eigenvalues of B are λ_i, $1 \le i \le d$, with $|\lambda_1| \ge \cdots \ge |\lambda_d| > r > 1$. Then the eigenvalues of A^n are λ_i^{-n}, $1 \le i \le d$. Using Lemma 2.1 we can find n large enough so that $\|A^n\| \le c \cdot r^{-n} < 1$ for some constant $c > 0$ independent of

n. It follows that for each $\mathbf{s} \in \mathbb{S}^n$, $f_{\mathbf{s}}$ is an affine contraction with corresponding matrix $A^{|\mathbf{s}|} = A^n$. Note that $\sum_{\mathbf{s}\in\mathbb{S}^n} \left|\det A^{|\mathbf{s}|}\right| = N^n \cdot N^{-n} = 1$ and hence Theorem 1.2 implies the corollary immediately □

Proof of Corollary 1.5. Let $A = B^{-1}$. Then $f_i(\mathbf{x}) = w_i A\mathbf{x} + \mathbf{a}_i$, $\mathbf{x} \in \mathbb{R}^d$, $i = 1, \cdots, N$. For any $n \in \mathbb{N}$, we have $K = \bigcup_{\mathbf{s}\in\mathbb{S}^n} f_{\mathbf{s}}(K)$, where

$$f_{\mathbf{s}}(\mathbf{x}) = w_{s_1} A \cdot w_{s_2} A \cdots w_{s_n} A\mathbf{x} + w_{s_1} A \cdots w_{s_{n-1}} A\mathbf{a}_{s_n} \cdots + w_{s_1} A\mathbf{a}_{s_2} + \mathbf{a}_{s_1}.$$

By $B\mathbb{W} = \mathbb{W}B$, it is easy to see that for any $w_i \in \mathbb{W}$, $Aw_i = w_j A$ for some $w_j \in \mathbb{W}$. Hence $w_{s_1} A \cdot w_{s_2} A \cdots w_{s_n} A = wA^n$ for some $w \in \mathbb{W}$. It follows that $\|w_{s_1} A \cdot w_{s_2} A \cdots w_{s_n} A\| \le \left(\max_{w\in\mathbb{W}} \|w\|\right) \|A^n\|$. Now the rest is the same as in the proof of Corollary 1.4. □

Proof of Theorem 1.6. To prove this theorem, we need to use a result in [12] and a similar proof as in Theorem 1.2. For $E \subset \mathbb{R}^d$ and $\varepsilon > 0$, let

$$U(\varepsilon, E) = \{\mathbf{x} : |\mathbf{x} - \mathbf{y}| < \varepsilon \text{ for some } \mathbf{y} \in E\}.$$

For $\mathbf{s} \in \mathbb{S}^*$, define $G_{\mathbf{s}} = U(\varepsilon r_{\mathbf{s}}, K_{\mathbf{s}})$. Then from [12] we know that there exists a $\mathbf{k} \in \mathbb{S}^*$ such that for some small $\varepsilon > 0$, the set $O = G_{\mathbf{k}} \cup \left(\bigcup_{\mathbf{j}\in\mathbb{S}^*} G_{\mathbf{jk}}\right)$ is an OSC set for $\{f_i\}_{i=1}^N$. Suppose $|\mathbf{k}| = m$. Let $E = K \cap O$ and $F = K \backslash E$. For $l \in \mathbb{N}$, let

$$J = \left\{\mathbf{s} = (\mathbf{s}_1, \cdots, \mathbf{s}_j, \cdots, \mathbf{s}_l) \in \mathbb{S}^{ml} : \mathbf{s}_j = \mathbf{k} \text{ for some } j\right\} \text{ and } L = \mathbb{S}^{ml} \backslash J$$

as in the proof of Theorem 1.2. Then

$$F = K \backslash O \subset K \backslash \bigcup_{\mathbf{j}\in J} G_{\mathbf{j}} \subset K \backslash \bigcup_{\mathbf{j}\in J} K_{\mathbf{j}} \subset \bigcup_{\mathbf{j}\in L} K_{\mathbf{j}}.$$

Since $\sum_{i=1}^N r^\alpha = 1$, it is easy to see that $\sum_{\mathbf{s}\in L} r_{\mathbf{s}}^\alpha = (1 - r_{\mathbf{k}}^\alpha)^l$. Let $\widetilde{r} = \max_{1\le i\le N} \{r_i\}$, $r = \min_{1\le i\le N} \{r_i\}$ and $\kappa = |K|$. Then we have, for $\mathbf{s} \in L$, $r^{lm}\kappa \le |K_{\mathbf{s}}| = r_{\mathbf{s}}\kappa \le \widetilde{r}^{lm}\kappa$. Let $\delta_l = \widetilde{r}^{lm}\kappa$. Then for $0 < \beta < \alpha$,

$$\mathcal{H}^\beta_{\delta_l}(F) \le \mathcal{H}^\beta_{\delta_l}\left(\bigcup_{\mathbf{j}\in L} K_{\mathbf{j}}\right) \le \sum_{\mathbf{s}\in L} |K_{\mathbf{s}}|^\beta = \sum_{\mathbf{s}\in L} (r_{\mathbf{s}}\kappa)^\beta \le \kappa^\beta \left(r^{m(\beta-\alpha)}(1 - r_{\mathbf{k}}^\alpha)\right)^l.$$

Hence by the similar arguments as in the proof of Theorem 1.2, we have $\dim_H(F) \le \beta < \alpha$ if

$$\beta = \alpha - \frac{\ln(1 - r_{\mathbf{k}}^\alpha)}{m \ln r}.$$

Now suppose $i \ne j$. Since $K = E \cup F$, it is clear that

$$f_i(K) \cap f_j(K) \subset (f_i(E) \cap f_j(E)) \cup f_i(F) \cup f_j(F).$$

But O is an OSC set and $E \subset O$, so $f_i(E) \cap f_j(E) \subset f_i(O) \cap f_j(O) = \emptyset$. Hence $f_i(K) \cap f_j(K) \subset f_i(F) \cup f_j(F)$. Since f_i and f_j are similarities, $\dim_H(f_i(F)) = \dim_H(f_j(F)) = \dim_H(F) < \alpha$. Therefore $\dim_H(f_i(K) \cap f_j(K)) \le \dim_H(F) < \alpha$. □

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