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Iterated Function System and Ruelle Operator

Ai Hua Fan*

Department of Mathematics, University of Cergy-Pontoise, 2, Avenue A. Chauvin, 95302 Cergy-Pontoise, France and Institute of Mathematical Sciences, The Chinese University of Hong Kong, Hong Kong
E-mail: fan@math.pst.u-cergy.fr

and

Ka-Sing Lau

Department of Mathematics, The Chinese University of Hong Kong, Hong Kong and Department of Mathematics, University of Pittsburgh, Pittsburgh, Pennsylvania 15260
E-mail: kslau@math.cuhk.edu.hk and lauks + @pitt.edu

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We consider a generalized iterated function system where the weights are variable functions. By using the Ruelle operator and a dynamical system consideration we prove that if the system is contractive and the weights are strictly positive functions and satisfy the Dini condition, then there exists a unique eigenmeasure (corresponding to the Ruelle operator) on the attractor. If in addition the maps are conformal and satisfy the open set condition, then we prove that they satisfy the strong open set condition, and by using this we can give a description of the L^p-scaling spectrum and the multifractal structure of the eigenmeasure. The work extends some results of [*Proc. London Math. Soc.* **73** (1996), 105–154; *Adv. Appl. Math.* **19** (1997), 486–513; *J. Statist. Phys.* **86** (1997), 233–275; *Indiana Univ. Math. J.* **42** (1993), 367–411].

INTRODUCTION

Let (X, d) be a compact metric space, let $\{w_j\}_{j=1}^m$ be a set of contractive maps from X into X, and let $\{p_j\}_{j=1}^m$ be a set of nonnegative continuous functions on X. We call the triple $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ a *contractive system*.

* New address: Department of Mathematics, University of Pícardie, 33, Rue Saint Leu, 80039 Amiens Cedex 1, France.

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This setup is a generalization of the usual iterated function system with constant probability weights or with variable weights $\{p_j\}_{j=1}^m$ satisfying the normalization condition $\sum_{j=1}^m p_j(x) = 1$ [1, 12]. In this paper, we are interested in the probability measures which are solutions of the equation,

$$\lambda\mu = \sum_{j=1}^{m} p_j(x)\,\mu \circ w_j^{-1}. \tag{0.1}$$

For this we consider the Ruelle operator T: $C(X) \to C(X)$ where $C(X)$ is the space of continuous functions on X and

$$Tf(x) = \sum_{j=1}^{m} p_j\big(w_j(x)\big) f\big(w_j(x)\big). \tag{0.2}$$

The adjoint operator T^* on the space of regular Borel measures $M(X)$ is given by

$$T^*\mu = \sum_{j=1}^{m} p_j(x)\,\mu \circ w_j^{-1}.$$

It follows that the measure solution in (0.1) can be regarded as an eigenvalue problem of the adjoint operator T^*. We are particularly interested in the largest eigenvalue ρ, which corresponds to the spectral radius of T^*. We call $\log \rho$ the *pressure* of the system. The eigenmeasure associated with ρ, when suitably normalized, is called the *Gibbs measure*. Our first goal in this paper is to establish the fundamental existence and uniqueness of the eigenmeasures under the condition that the p_js are strictly positive and $\log p_j$ satisfy the Dini condition. In general, the existence is quite easy to obtain, but the uniqueness is more intricate. The key idea of the proof is to relate this with a dynamical system on a symbolic space. Note that in [20], Quas gave an example that the eigenmeasure is not unique if we just assume positivity and continuity on the p_js. In [9], the existence and uniqueness of the eigenmeasure was discussed when the maps w_js were just weakly contractive.

We remark that an alternative approach to this eigenproblem is to use the theory of quasi-compact operators [11]. But then the Dini condition does not work and the stronger Hölder condition is needed. In [1], there is a probabilistic proof of the theorem but under a normalization hypothesis $\sum p_j(x) = 1$ for all $x \in X$. We see that even if we would like to study the system with the hypothesis $\sum p_j(x) = 1$, we are yet led to consider the others systems with weights for which the hypothesis is no longer valid. Strichartz, Taylor, and Zhang [26] have also studied the problem here on

$X = [0, 1]$ using continuous w_js only, but with a certain separation condition. A more detailed study along this line is given by Öberg [17].

If $\{w_j\}_{j=1}^m$ are contractive conformal maps on a domain of $\mathbf{R}^d$, we call the corresponding eigenmeasure μ a *self-conformal measure*. We study the multifractal formalism and the L^q-scaling spectrum $\tau(q)$ of such measure by making use of the Ruelle operator of the form,

$$T_{q,\alpha} f(x) = \sum_{j=1}^{m} p_j\big(w_j(x)\big)^q |w_j'(x)|^{-\alpha} f\big(w_j(x)\big),$$

where $|w_j'(x)|$ is the operator norm of $w_j'(x)$ on $\mathbf{R}^d$, and equals $|\det w_j'(x)|^{1/d}$ because of the conformal property. Under the condition that $\{w_j\}_{j=1}^m$ satisfies the *open set condition* (see the definition in Section 2) and that $\{\log p_j\}_{1 \le j \le m}$ satisfies the Dini condition, we prove that the L^q-scaling spectrum $\tau(q)$ is the unique α such that the spectral radius of $T_{q,\alpha}$ equals 1 (Theorem 3.3). The function $\tau(q)$ is strictly convex and analytic. Its Legendre transformation describes the multifractal structure of the self-conformal measure (Theorem 3.4). In the case that the weights are constant probability weights and the w_j's are similarities with contracting ratio r_j, it is well known that $\tau(q) = \alpha$ satisfies

$$\sum_{j=1}^{m} p_j^q r_j^{-\alpha} = 1,$$

which can be viewed as a special form of $T_{q,\alpha} 1 = 1$.

The proofs of the previous results on $\tau(q)$ and on the multifractal formalism are based on the so-called *measure separated property* in the sense that $\mu(w_i(K) \cap w_j(K)) = 0$, $i \neq j$. By using the technique of Schief [24] and Lau and Wang [13], we show that the open set condition implies the *strong* open set condition (Lemma 2.6) and hence the measure separated property (Theorem 2.2). This property allows us to establish a one-to-one relationship between the attractor K and a symbolic space Σ except for a μ-zero set. Then the dynamical system on the symbolic space applies.

The self-conformal measures have also been studied by Strichartz [25], Mauldin and Urbański [16], Pesin and Weiss [19], and Patzschke [18]. The technique in [25] is to use approximation by family of self-similar measures while the other three make use of the Ruelle operator. In [16], the setup is for iterated function systems with a countable family of conformal maps and the main interest is on the structure of the attractor. In [19], the consideration is on the conformal repelling system determined by a conformal map g on X (the branches of g^{-1} correspond to the contractive system), and a Moran-like iteration under a separation condition on the attractor which is stronger than the open set condition we use here. The separation assumption in [18] is a consequence of the open set condition as is proved in Lemma 2.6.

The material of the paper is organized as follows. In Section 1, we use the Ruelle–Perron–Frobenius theorem on the symbolic space to prove the existence and uniqueness of the Gibbs measure for $(X, \{w_j\}, \{p_j\})$ under the assumption that the weight functions $\log p_j$ satisfy the Dini condition. We also establish the Gibbs property under the measure separation property. In Section 2, we study the conformal iterated function system with the open set condition and we prove that for such a system the measure separation property holds (Theorems 2.1 and 2.2). We also consider the Hausdorff measure $\mathscr{H}^s$ on the attractor as a self-conformal measure. This result is known [16] but the proof here is quite natural under the present setup and the assumption is slightly more general. In Section 3, we consider the L^q-scaling spectrum and we prove the multifractal formalism for the self-conformal measures (Theorems 3.3 and 3.4).

1. TRANSFER OPERATOR AND GIBBS PROPERTY

Let $\{w_j\}_{j=1}^m$ be a contractive iterated system on X. It is well known that there exists a compact attractor K that satisfies $K = \bigcup_{j=1}^m w_j(K)$ [10]. This invariance property allows us to restrict the operator T on $C(K)$ and T^* on $M(K)$. For a function $p\colon X \to \mathbf{R}$, we denote its modulus of continuity by $\Omega(p, t) = \max\{|p(x) - p(y)|\colon d(x, y) \le t\}$.

THEOREM 1.1. *Suppose $(X, \{w_j\}, \{p_j\})$ is a contractive system such that*

$$\int_0^1 \frac{\Omega(\log p_j, t)}{t}\, dt < \infty, \qquad 1 \le j \le m.$$

Let K be the attractor and let ρ be the spectral radius of T restricted to $C(K)$. Then there exists a unique $0 < h \in C(K)$ and a unique probability measure $\mu \in M(K)$ such that

$$Th = \rho h, \qquad T^*\mu = \rho\mu, \qquad \langle \mu, h \rangle = 1.$$

Moreover, for every $f \in C(K)$, $\rho^{-n} T^n f$ converges uniformly to $\langle \mu, f \rangle h$, and for every $\xi \in M(K)$, $\rho^{-n} T^{*n} \xi$ converges weakly to $\langle \xi, h \rangle \mu$.

The measure $\tilde{\mu} = h\mu$ is called the *Gibbs measure* of the system. The condition of the modulus of continuity on $\log p_i$ is called the *Dini condition*. The special case for symbolic spaces is known as the Ruelle–Perron–Frobenius Theorem [3, 8, 23]. We are going to prove Theorem 1.1

as a consequence of this special case. Let us first introduce some notations. For a multi-index $J = (j_1, j_2, \dots, j_n)$ with $j_k \in \{1, 2, \dots, m\}$, let $|J| = n$ denote the length of J and let

$$\begin{aligned} w_J &= w_{j_1} \circ w_{j_2} \circ \cdots \circ w_{j_n}, \\ K_J &= w_J(K), \\ p_J(x) &= p_{j_1}(x) \cdots p_{j_n}(x), \\ p_{w_J}(x) &= p_{j_1}(w_{j_1} \circ \cdots \circ w_{j_n} x) \cdots p_{j_{n-1}}(w_{j_{n-1}} \circ w_{j_n} x) p_{j_n}(w_{j_n} x). \end{aligned}$$

It follows by induction that

$$T^n f(x) = \sum_{|J| = n} p_{w_J}(x) f(w_J(x)).$$

The strictly positive eigenfunction h in Theorem 1.1 will play a crucial role. It allows us to introduce the normalization of $\tilde{T}: C(K) \to C(K)$ defined by

$$\tilde{T}f = \frac{1}{\rho h} T(hf).$$

The following proposition is easy to check.

PROPOSITION 1.2. *Let h be the strictly positive eigenfunction of T and let*

$$a_j(x) = \frac{p_j(w_j x) h(w_j x)}{\rho h(x)}.$$

Then

$$\tilde{T}f(x) = \sum_{j=1}^{m} a_j(x) f(w_j x),$$

and $\tilde{T}1 = 1$ (i.e., $\sum_{j=1}^m a_j(x) = 1$).

The key idea of proving Theorem 1.1 is to establish a "conjugacy" relation between our system and a symbolic space. By a symbolic space we mean the infinite product space $\Sigma = \{1, 2, \dots, m\}^{\mathbf{N}}$. For $\sigma = (\sigma_n) \in \Sigma$, we write $\sigma|_k = (\sigma_1, \dots, \sigma_k)$ and $\sigma|^k = (\sigma_{k+1}, \sigma_{k+2}, \dots)$. The shift transformation on Σ is defined by $\theta(\sigma) = \sigma|^1$. For σ and σ', we define their distance as $d(\sigma, \sigma') = e^{-n(\sigma, \sigma')}$ where $n(\sigma, \sigma')$ is the largest n such that

$\sigma|_n = \sigma'|_n$. It follows that a cylinder set $I_n(\sigma)$ is the ball of radius e^{-n} of center σ. Define

$$u_j\colon \Sigma \to \Sigma, \quad \text{by } u_j\sigma = j\sigma,\ 1 \le j \le m.$$

Then $\theta^{-1}(\sigma) = \{u_j(\sigma)\}$. The system $(\Sigma, \{u_j\}, \{q_j\})$ with an arbitrary choice for q_j is called a *symbolic system*. With a suitably defined weight q_j, this symbolic system becomes a prototype for a general system. For our case we define $q\colon \Sigma \to \mathbf{R}^+$ by $q(\sigma) = q_j(\sigma) = p_j(\pi(\sigma))$ if $\sigma \in u_j(\Sigma)$ where π is defined in the next proposition. Let h_Σ and ν be the eigenfunction and eigenmeasure of the system $(\Sigma, \{u_j\}, q)$ as in Theorem 1.1. The *Gibbs measure* of this system will be denoted by $\tilde{\nu} = h_\Sigma \nu$. The following establishes the conjugacy and ensures the existence of μ in Theorem 1.1.

PROPOSITION 1.3. *Let $(X, \{w_j\}, \{p_j\})$ be a contractive system with attractor K. Let $y \in K$ be fixed and let $\pi\colon \Sigma \to K$ be defined by*

$$\pi(\sigma) = \lim_{n\to\infty} w_{\sigma|_n}(y) = \lim_{n\to\infty} w_{\sigma_1} \cdots w_{\sigma_n}(y).$$

(i) *The limit exists and is independent of $y \in K$. The mapping π is continuous and onto, and satisfies $\pi \circ u_j = w_j \circ \pi$, $1 \le j \le m$.*

(ii) *Let μ be the image of ν under π, then μ satisfies $T^*\mu = \rho\mu$.*

Proof. (i) is a consequence of the contractivity of w_j and is well known [6]. To prove (ii) we define the transfer operator $S\colon C(\Sigma) \to C(\Sigma)$ by

$$Sg(\sigma) = \sum_{j=1}^{m} q(u_j\sigma)g(u_j\sigma),$$

and also $\tau\colon C(K) \to C(\Sigma)$ by $\tau f = f \circ \pi$. By observing that for $x = \pi(\sigma)$ and $1 \le j \le m$,

$$q(u_j\sigma) = p_j\big(\pi \circ u_j(\sigma)\big) = p_j\big(w_j \circ \pi(\sigma)\big) = p_j(w_j x),$$

it is routine to check that $\tau T = S\tau$. It follows that $S^n 1(\sigma) = T^n 1(\pi\sigma)$. Because T and S are positive operators, their spectral radii are, respectively, given by

$$\lim_{n\to\infty} \|T^n 1\|^{1/n} \quad \text{and} \quad \lim_{n\to\infty} \|S^n 1\|^{1/n}.$$

From this we conclude that T and S have the same spectral radius ρ. Now suppose $S^*\nu = \rho\nu$ and μ is the image under π of ν. Then $T^*\mu = \rho\mu$ is checked by

$$\langle \rho\mu, f \rangle = \langle \rho\nu, f \circ \pi \rangle = \langle S^*\nu, \tau f \rangle = \langle \nu, S\tau f \rangle$$
$$= \langle \nu, \tau Tf \rangle = \langle \mu, Tf \rangle = \langle T^*\mu, f \rangle.$$

∎

Proof of Theorem 1.1. For a multi-index J, we can define, analogous to p_{w_J},

$$q_{u_J}(\sigma) = q\big(u_{j_1} \cdots u_{j_n}(\sigma)\big) \cdots q\big(u_{j_{n-1}} u_{j_n}(\sigma)\big) q\big(u_{j_n}(\sigma)\big).$$

Then for $x = \pi(\sigma)$, we have $q_{u_J}(\sigma) = p_{w_J}(x)$. Note that $\sigma = u_{\sigma|_n}(\theta^n\sigma)$. Then for σ and σ' in Σ such that $\sigma|_n = \sigma'|_n$,

$$\begin{aligned}\log q(\sigma) &- \log q(\sigma') \\ &= \log p_{\sigma_1}\big(\pi \circ u_{\sigma|_n}(\theta^n\sigma)\big) - \log p_{\sigma_1}\big(\pi \circ u_{\sigma|_n}(\theta^n\sigma')\big) \\ &= \log p_{\sigma_1}\big(w_{\sigma|_n} \circ \pi(\theta^n\sigma)\big) - \log p_{\sigma_1}\big(w_{\sigma|_n} \circ \pi(\theta^n\sigma')\big).\end{aligned}$$

It follows that

$$\Omega(\log q, e^{-n}) \leq \max_{1 \leq j \leq m} \Omega\big(\log p_j, (r_{\max})^n\big),$$

where $r_{\max} < 1$ is the maximum of the contractive ratios of the w_j's. Note that the Dini condition implies that for any $0 < a < 1$,

$$\sum_{n=1}^{\infty} \Omega\big(\log p_j, a^n\big) < \infty.$$

The last two inequalities imply that the Dini condition of the Ruelle–Perron–Frobenius theorem is satisfied for the system $(\Sigma, \{u_j\}, q)$ (see [8]). Hence there exists a strictly positive function $h_\Sigma \in C(\Sigma)$ and a probability measure $\nu \in M(\Sigma)$ such that

$$Sh_\Sigma = \rho h_\Sigma, \qquad S^*\nu = \rho\nu, \qquad \langle \nu, h_\Sigma \rangle = 1.$$

Now note that for any $f \in C(K)$ and for any $n \geq 1$,

$$(T^n f) \circ \pi = S^n(f \circ \pi),$$

and that $\rho^{-n}S^n(f \circ \pi)$ converges uniformly. Because π is a mapping from Σ *onto* K, then $\rho^{-n}T^n f$ converges uniformly to a function, which is an eigenfunction associated with ρ if the limit is not zero. Actually, if we take f to be the constant function equal to 1, the limit of $\rho^{-n}S^n 1$ is the function

h_Σ which is strictly positive. We can deduce that the corresponding limit of $\rho^{-n}T^n 1$ is also strictly positive. Thus we obtain a strictly positive eigenfunction of T associated with ρ. Take such a strictly positive eigenfunction h such that $\langle \mu, h \rangle = 1$. We claim $h \circ \pi = h_\Sigma$. This is because $\langle \nu, h \circ \pi \rangle = \langle \mu, h \rangle = 1$ and

$$S(h \circ \pi) = S(\tau h) = \tau(Th) = \rho \tau h = \rho(h \circ \pi).$$

The claim follows then from the uniqueness of h_Σ. Let $\tilde{S}$ and $\tilde{T}$ be defined as is Proposition 1.2. Note that

$$\begin{aligned}(\tau \tilde{T}^n) f &= \frac{1}{\rho^n h \circ \pi} (\tau T^n)(hf) \\ &= \frac{1}{\rho^n h \circ \pi} S^n \tau(hf) = \frac{1}{\rho^n h_\Sigma} S^n (h_\Sigma \cdot \tau f) = \tilde{S}^n(\tau f).\end{aligned}$$

Because $\tilde{S}^n(\tau f)$ converges uniformly to the constant $\langle \nu, \tau f \rangle$, we deduce that $\tilde{T}^n f$ also converges uniformly to the same constant, using again the surjectivity of π and the previous argument. The constant is actually $\langle \mu, f \rangle$. Thus we have proved the convergence of $\rho^{-n}T^n f$ to $\langle \mu, f \rangle h$.

For any $\xi \in M(K)$,

$$\langle \rho^{-n} T^{*n} \xi, f \rangle = \langle \xi, \rho^{-n} T^n f \rangle \to \langle \xi, h \rangle \langle \mu, f \rangle = \langle \langle \xi, h \rangle \mu, f \rangle,$$

which means $\rho^{-n} T^{*n} \xi$ converges weakly to $\langle \xi, h \rangle \mu$.

For the uniqueness of μ, we suppose that there exists another eigenmeasure μ'. Then

$$\mu' = \rho^{-n} T^{*n} \mu' \to \langle \mu', h \rangle \mu.$$

Because μ and μ' are probability measures, we have

$$1 = \langle \mu', 1 \rangle = \langle \mu', h \rangle \langle \mu, 1 \rangle = \langle \mu', h \rangle.$$

Hence $\mu' = \mu$. The uniqueness of h is a consequence of the convergence $\rho^{-n}T^n f$, using the preceding argument. ∎

PROPOSITION 1.4. *Let $(X, \{w_j\}, \{p_j\})$ be defined as in Theorem* 1.1. *For $q \in R$, consider the Ruelle operator T_q: $C(K) \to C(K)$ defined by*

$$T_q f(x) = \sum_{j=1}^{m} p_j\big(w_j(x)\big)^q f\big(w_j(x)\big).$$

Then the pressure function $\log \rho(q)$, *where $\rho(q)$ is the spectral radius of T_q, is a real analytic function.*

Proof. The proposition is well known for $S_q: C(\Sigma) \to C(\Sigma)$ when the p_js are strictly positive Hölder continuous functions (see, for example, [23]). Its proof depends on that $\rho(q)$ is an isolated point of the spectrum of T_q and that the eigenspace is one dimensional ([5, p. 587]). The same proof will apply here. ∎

In the following we consider the important *Gibbs property* of the eigenmeasure in Theorem 1.1. To abbreviate notations we write $a_n \approx b_n$ to mean the existence of a $C > 0$ so that $0 < C^{-1} a_n \le b_n \le C a_n$ for all n.

LEMMA 1.5. *Let* $\{a_n\}, \{b_n\}$ *be positive sequences such that* $0 < \eta \le a_n, b_n$ *and* $|a_n - b_n| \le r_n$ *with* $\sum r_n < \infty$, *then*

$$\prod_{j=1}^{n} a_j \approx \prod_{j=1}^{n} b_j.$$

Proof. It suffice to observe from the hypothesis that

$$1 - \frac{r_n}{\eta} \le \frac{a_n}{b_n} \le 1 + \frac{r_n}{\eta}.$$

∎

THEOREM 1.6. *Let* $(X, \{w_j\}, \{p_j\})$ *be a contractive system with attractor* K *as in Theorem* 1.1. *Suppose* $\mu(K_L \cap K_J) = 0$ *for all* $L \neq J$ *with* $|L| = |J|$. *Then*

(i) μ *has the Gibbs property*: *for each* $x \in K$ *and* J, $\mu(K_J) \approx \rho^{-|J|} p_{w_J}(x)$;

(ii) *For* μ *almost all* $x \in K$ *and for* σ *such that* $x = \pi(\sigma)$, *we have*

$$\lim_{n \to \infty} \frac{\log \mu(K_{\sigma|_n})}{n} = \sum_{j=1}^{m} \int_{K_j} (\log p_i(x)) h(x)\, d\mu(x) - \log \rho.$$

Proof. Let ν be the corresponding eigenmeasure for the symbolic space, it is known [3] that

$$\nu(I_n(\sigma)) \approx \rho^{-n} \prod_{j=1}^{n} q(\theta^{j-1}\sigma). \tag{1.1}$$

Note that for any $1 \le j \le n$,

$$q(\theta^{j-1}\sigma) = p_{\sigma_j}(\pi(\theta^{j-1}\sigma)) = p_{\sigma_j}\left(w_{\sigma_j} \cdots w_{\sigma_n}(\pi\sigma|^n)\right).$$

For any $x, y \in K$,

$$\left|\log p_{\sigma_j}\left(w_{\sigma_j} \cdots w_{\sigma_n}(x)\right) - \log p_{\sigma_j}\left(w_{\sigma_j} \cdots w_{\sigma_n}(y)\right)\right| \leq \max_{1 \leq j \leq m} \Omega\left(\log p_j, (r_{\max})^n\right),$$

where $r_{\max}$ is the largest of the contractive ratios of w_js. The Dini condition implies that $\sum_{k=1}^{\infty} \Omega(\log p_j, a^k) < \infty$ for any $0 < a < 1$ and $1 \leq j \leq m$. By the foregoing lemma we have for $x = \pi(\sigma) \in K$,

$$\prod_{j=1}^{n} q(\theta^{j-1}\sigma) = p_{w_{\sigma|_n}}(\pi\sigma|^n) \approx p_{w_{\sigma|_n}}(x). \tag{1.2}$$

Next we claim that $\mu(K_{\sigma|_n}) = \nu(I_n(\sigma))$ which implies (i) by (1.1) and (1.2). In fact because μ is the image of ν under π, we have $\mu(K_J) = \nu(\pi^{-1}(K_J))$ for any J. Note that

$$I_J \subset \pi^{-1}(K_J) \subset I_J \cup \left(\bigcup_{|L|=|J|,\, L \neq J} \pi^{-1}(K_L \cap K_J)\right).$$

But $\nu(\pi^{-1}(K_L \cap K_J)) = 0$ by hypothesis, it follows that $\mu(K_J) = \nu(I_J)$.

To prove (ii) we let

$$H = K \setminus \bigcup_{n=1}^{\infty} \{K_L \cap K_J \colon |L| = |J| = n,\, L \neq J\}.$$

Then by assumption $H = K$ except for a μ-zero set. For each $x \in H$ there exists a unique $\sigma \in \Sigma$ such that $x = \pi(\sigma)$. We note that the Gibbs measure $\tilde{\nu}$ is invariant and ergodic. So by the ergodic theorem, for ν-almost all σ,

$$\lim_{n \to \infty} \frac{\log \nu(I_n(\sigma))}{n} = \int_{\Sigma} \log q(\sigma)\, d\tilde{\nu}(\sigma) - \log \rho.$$

To end the proof of (ii), it suffices to calculate the last integral. Let $\nu(j, \cdot)$ be the image of $\nu|_{\Sigma_j}$ under the mapping $u_j^{-1}\colon \Sigma_j \to \Sigma$ (Σ_j being defined like K_j). Let $\mu(j, \cdot)$ be the image of $\mu|_{\Sigma_j}$ under the mapping $w_j^{-1}\colon K_j \to K$. These mean that

$$\int_K f(x)\, d\mu(j, x) = \int_{K_j} f\left(w_j^{-1}x\right) d\mu(x), \qquad (\forall f \in C(K)),$$

$$\int_{\Sigma} \varphi(\sigma)\, d\nu(j, \sigma) = \int_{\Sigma_j} \varphi\left(u_j^{-1}\sigma\right) d\nu(\sigma), \qquad (\forall \varphi \in C(\Sigma)).$$

Note that u_j^{-1} is the restriction of θ on Σ_j. Note also that $\mu(j,\cdot)$ is the image of $\nu(j,\cdot)$ under π. That means

$$\int_K f(x)\,d\mu(j,x) = \int_\Sigma f(\pi\sigma)\,d\nu(j,\sigma), \qquad (\forall f \in C(K)).$$

Now we calculate

$$\begin{aligned}\int_\Sigma \log q(\sigma)\,d\tilde{\nu}(\sigma) &= \sum_j \int_{\Sigma_j} \big(\log p_j(\pi\sigma)\big) h(\pi\sigma)\,d\nu(\sigma)\\ &= \sum_j \int_\Sigma \Big(\log p_j\big(\pi u_j^{-1}\sigma\big)\Big) h\big(\pi u_j^{-1}\sigma\big)\,d\nu(j,\sigma)\\ &= \sum_j \int_\Sigma \Big(\log p_j\big(w_j^{-1}\pi\sigma\big)\Big) h\big(w_j^{-1}\pi\sigma\big)\,d\nu(j,\sigma)\\ &= \sum_j \int_K \Big(\log p_j\big(w_j^{-1}x\big)\Big) h\big(w_j^{-1}x\big)\,d\mu(j,\sigma)\\ &= \sum_j \int_{K_j} \big(\log p_j(x)\big) h(x)\,d\mu(x).\end{aligned}$$

∎

Note that $K_i \cap K_j = \emptyset$ for $i \neq j$ clearly satisfies the *measure separation condition* $\mu(K_J \cap K_L) = 0$. However it appears to be too strong (the system with this property can actually be identified with the symbolic space), a simple example that such a separation condition cannot be satisfied is the three similarities that generates the Sierpinski triangle. In the next section we impose more restrictions on the contractions w_j's to ensure the measure separation condition, namely, the open set condition and the conformality, which includes the Sierpinski triangle. Also we remark that if the condition holds and if the w_js are one to one, we can set up a topological conjugacy between H and $\pi^{-1}(H)$ and we can establish the following commutative diagram except for a μ-zero set,

$$\begin{array}{ccc} \Sigma & \underset{\theta}{\overset{\{u_j\}}{\rightleftarrows}} & \Sigma \\ \pi\downarrow & & \downarrow\pi' \\ K & \underset{\Theta}{\overset{\{w_j\}}{\rightleftarrows}} & K \end{array}$$

where $\Theta x = w_j^{-1}x$ if $x \in K_j \setminus \bigcup_{n=1}^{\infty}\{K_L \cap K_J\colon L \neq J,\ |L| = |J| = n\}$, and Θx is arbitrary for the rest of the x. This is used to induce the multifractal structure of the symbolic system to $(X, \{w_i\}, \{p_i\})$ (see Section 3).

2. SELF-CONFORMAL MEASURES AND OPEN SET CONDITION

We call a $\mathbf{R}^d$-valued map w defined on some open set $V \subset \mathbf{R}^d$ a conformal map if w is continuously differentiable and if $w'(x)$ is a self-similar matrix for each $x \in V$ (i.e., $w'(x)$ is a constant multiple of a rotation). Throughout the next two sections, we make the following assumptions on $\{w_j\}_{j=1}^m$:

(I) X is a compact subset of $\mathbf{R}^d$, w_j: $X \to X$ is one-to-one and contractive;

(II) w_j is conformal on an open set $V \supset X$;

(III) $\log |w_j'(x)|$ satisfies the Dini condition on X.

For such iterated function systems, the eigenmeasure in Theorem 1.1 is called a *self-conformal measure*. It is the probability measure μ satisfying

$$\rho\mu = \sum_{i=1}^{m} p_i(x)\,\mu \circ w_i^{-1},$$

where ρ is the spectral radius of T. The contractive ratio of w_j at x is the operator norm of the matrix $w_j'(x)$ on $\mathbf{R}^d$, denoted by $|w_j'(x)|$. The assumptions imply that $0 < |w_j'(x)| < 1$ and $|w_j'(x)|^d = |\det w_j'(x)|$. Note that by the chain rule,

$$w_J'(x) = w_{j_1}'(w_{j_2} \cdots w_{j_n} x) w_{j_2}'(w_{j_3} \cdots w_{j_n} x) \cdots w_{j_n}'(x).$$

We say that $\{w_j\}_{j=1}^m$ satisfies the *open set condition* (OSC) if there exists a bounded open set U such that $\overline{U} \subseteq V$ and

$$w_i(U) \subseteq U \quad \text{and} \quad w_i(U) \cap w_j(U) = \emptyset, \qquad i \neq j.$$

We call such U a *basic open set*. Let s be the (positive) number such that the Ruelle operator,

$$T_s f(x) = \sum_{j=1}^{m} |w_j'(x)|^s f(w_j x) \tag{2.1}$$

has spectral radius 1. The existence and uniqueness of such s is because the spectral radius $\rho(s) = \lim_{n\to\infty} \|T_s^n 1\|^{1/n}$ is continuous, strictly increasing with $\rho(0) = m$ and $\rho(\infty) = 0$ (note that $T_s^n 1(x) = \sum_{|J|=n} |w_J'(x)|^s$). Also note that (2.1) can be adjusted to (0.2) by writing $w_j'(x) = w_j'(w_j^{-1}(w_j(x)))$, so that $\lim_{n\to\infty} T_s^n 1(x) = h(x)$ uniformly for some $h > 0$ satisfying $T_s h = h$ (Theorem 1.1).

If U is a basic open set, it is easy to see that $K \subseteq \overline{U}$ so that μ is supported by $\overline{U}$. In all applications, it is important to have $K \subset U$ and μ is supported by U. However sometimes a basic open set U may not support μ. For example, consider the standard Cantor measure μ defined by

$$S_1 x = \frac{x}{3}, \qquad S_2 x = \frac{x}{3} + \frac{2}{3}.$$

If C denotes the Cantor set, then $U = (0,1) \setminus C$ is a basic open set but $\mu(U) = 0$ and $\mu(\partial U) = \mu(C) = 1$. Our main results in this section are the following two theorems.

THEOREM 2.1. *Suppose the contractive conformal maps* $\{w_j\}_{j=1}^m$ *satisfy the OSC. Then we can choose a basic open set U which supports all the self-conformal measures* μ *with weight function* $\{\log p_j\}_{j=1}^m$ *satisfying the Dini condition as in Theorem* 1.1.

THEOREM 2.2. *Let* $\{w_j\}$ *and* μ *be as in the previous text and let K be the attractor, then* $\mu(K_J \cap K_L) = 0$ *for* $|J| = |L|$, $J \neq L$, *and Theorem* 1.6 *holds for such* μ.

We need a few lemmas to prove the two theorems.

LEMMA 2.3. *Let K be the attractor.*

(i) *There exists a constant* C_1 *such that for any* $x, y \in K$,

$$|w_J'(x)| \leq C_1 |w_J'(y)|.$$

(ii) *There exist constants* C_2 *and* $\delta > 0$ *such that for* $x, y, z \in K$, $|x - y| \leq \delta$,

$$C_2^{-1} |w_J'(z)| \leq \frac{|w_J(x) - w_J(y)|}{|x - y|} \leq C_2 |w_J'(z)|.$$

Proof. Without loss of generality, we can assume that $\operatorname{diam} K \leq 1$. Let $a = \max_j \max_x |w_j'(x)|$. Then the Dini condition implies that $\sum_{k=1}^{\infty} \Omega(\log |w_j'|, a^k) < \infty$ $(1 \leq j \leq m)$. Hence for $J = j_1 j_2 \cdots j_n$, we have

$$\begin{aligned} \left| \log \frac{|w_J'(x)|}{|w_J'(y)|} \right| &= \sum_{k=1}^{n} \left| \log \left| w_{j_k}'\big(w_{j_{k+1}} \cdots w_{j_n}(x)\big) \right| - \log \left| w_{j_k}'\big(w_{j_{k+1}} \cdots w_{j_n}(y)\big) \right| \right| \\ &\leq \sum_{k=1}^{n} \sup_{1 \leq j \leq m} \left\{ \left| \log \left| w_j'(u) \right| - \log \left| w_j'(v) \right| \right| : |u - v| \leq a^k \right\} \\ &\leq \sum_{j=1}^{m} \sum_{k=1}^{\infty} \Omega\big(\log |w_j'|, a^k\big) < \infty. \end{aligned}$$

This implies (i). For (ii), we note that by the compactness of K and the openness of the domain V, there exists $\delta > 0$ such that the balls $B_\delta(x)$, $x \in K$ are all contained in V. Now suppose $x, y \in K$ with $|x - y| \le \delta$. Consider $g = h \circ \varphi$ with $h(u) = \sqrt{\sum_{j=1}^{d} |u_j|^2}$ and $\varphi(t) = w_J(tx + (1-t)y) - w_J(y)$. Note that h is differentiable at any $u \neq 0$ and $\varphi(t) \neq 0$ if $t \neq 0$. By applying the mean value theorem to g, we get

$$\left| w_J(x) - w_J(y) \right| = \left| w_J'(\xi)(x - y) \right|,$$

where ξ is the line segment joining x and y. The self-similar property of $w_J(x)$ yields

$$\left| w_J(x) - w_J(y) \right| = \left| w_J'(\xi) \right| |x - y|.$$

This together with (i) implies (ii). ∎

LEMMA 2.4. *Suppose* $\sum_{j=1}^{m} p_j(x)$ *for all* $x \in K$. *If* Λ *is a set of multi-indices such that the cylinder sets* $\{I_J\colon J \in \Lambda\}$ *form a finite disjoint cover of* Σ, *then we have*

$$\sum_{J \in \Lambda} p_{w_J}(x) = 1, \quad \textit{for all } x \in K. \tag{2.2}$$

Proof. Let $n = \max\{|J|\colon J \in \Lambda\}$. We have by induction,

$$\sum_{|J| = n} p_{w_J}(x) = 1, \quad \text{for all } x \in K.$$

Suppose now $J \in \Lambda$ with $|J| < n$, we can replace p_{w_J} by the sum $\sum_{j=1}^{m}$ because

$$\begin{aligned} \sum_{j=1}^{m} p_{w_{jJ}}(x) &= \sum_{j=1}^{m} p_j\big(w_j(w_J x)\big) p_{w_J}(x) \\ &= p_{w_J}(x) \sum_{j=1}^{m} p_j\big(w_j(w_J x)\big) = p_{w_J}(x). \end{aligned} \tag{2.3}$$

The replacement does not change the expression we are considering. Note also that for every J' with $|J'| = n$, it must come from one of the J in Λ. Continue this process and apply (2.1), the lemma follows. ∎

LEMMA 2.5. *Let* U *be a basic open set in the OSC and let* μ *be the self-conformal measure, then* μ *is either concentrated in* U *or in* ∂U, *the boundary of* U.

Proof. The proof is a modification of a proof in [13]. By Proposition 1.2, we can assume without loss of generality that $\sum_{j=1}^{m} p_j(x) = 1$ for all $x \in K$. Suppose $\mu(U) \neq 0$, then the self-conformality of μ implies that

$$\mu(U_J) = \int_K \sum_{|L|=|J|} p_{w_L}(x) 1_{U_J}(w_L x)\, d\mu(x).$$

We observe that $K_L \cap U_J = \varnothing$ when $J \neq L$, $|J| = |L|$. Otherwise, we have

$$\varnothing \neq K_L \cap U_J \subseteq \overline{U}_L \cap U_J,$$

which is impossible because U_J and U_L are open and disjoint. Consequently for such J and L, $1_{U_J}(w_L(x)) = 0$ for every $x \in K$. Then

$$\begin{aligned}
\mu\Big(\bigcup_{|J|=k} U_J\Big) &= \sum_J \mu(U_J) \\
&= \sum_J \int_K p_{w_J}(x) 1_{U_J}(w_J(x))\, d\mu(x) \\
&= \int_K \sum_J p_{w_J}(x) 1_U(x)\, d\mu(x) \\
&= \mu(U \cap K) = \mu(U).
\end{aligned}$$

(The fourth identity is because $\sum_J p_{w_J}(x) = 1$, and the last identity is because μ is concentrated in K.) It follows that

$$\begin{aligned}
\mu(\partial U) + \mu(U) &= \mu(\overline{U}) = 1 = \mu\Big(\bigcup_{|J|=k} \overline{U}_J\Big) \\
&= \mu\Big(\bigcup_{|J|=k} U_J\Big) + \mu\Big(\bigcup_{|J|=k} \partial U_J\Big) = \mu(U) + \mu\Big(\bigcup_{|J|=k} \partial U_J\Big).
\end{aligned}$$

Consequently we have

$$\mu(\partial U) = \mu\Big(\bigcup_{|J|=k} \partial U_J\Big).$$

By noting that $\partial U \cap K \subseteq \bigcup_{|J|=k}(\partial U_J \cap K)$ (see [13, Lemma 2.2(iv)] for the proof in the case of self-similarities, the proof is topological and hence works for the contractive invertible w_js here), we have

$$\mu\Big(\bigcup_{|J|=k} \partial U_J \setminus \partial U\Big) = 0.$$

Let $F = \bigcup_J \partial U_J \cup \partial U$. Then the earlier text implies that $\mu(F \setminus \partial U) = 0$ and hence $\mu(U \setminus F) = \mu(U \setminus \partial U) = \mu(U)$.

If $\mu(U) \neq 0$, we define $\lambda = a\mu|_{U \setminus F}$ where $a = \mu(U \setminus F)^{-1}$. Because $w_i^{-1}(U \setminus F) \cap \overline{U} \subseteq U \setminus F$ ([13, Lemma 2.2(iii)] we have

$$\lambda(A) = \sum_{i=1}^{m} p_{w_i}(x)\lambda\left(w_i^{-1}(A)\right),$$

for any Borel subset $A \subseteq U \setminus F$, F, and $X \setminus U$. Because λ is a probability measure, the uniqueness of the eigenmeasure implies that $\lambda = \mu$ and proves the lemma. ∎

For any two sets A, B in $\mathbf{R}^n$, we use $|A|$ to denote the diameter of A, and we let

$$D(A, B) = \inf\{d(x, y) \colon x \in A,\ y \in B\}.$$

be the distance of A and B. For a fixed $x \in K$, we let $r_J = |w_J'(x)|$. Then it follows from Lemma 2.3(i) and the chain rule that there exists $C > 0$ such that

$$\frac{1}{C} r_I r_J \leq r_{IJ} \leq C r_I r_J. \tag{2.4}$$

For small $t > 0$, let

$$\Lambda_t = \left\{J = (j_1, \dots, j_n) \colon n \text{ is the smallest such that } r_J < t\right\}.$$

LEMMA 2.6. *Suppose the contractive conformal family* $\{w_j\}_{j=1}^m$ *satisfies the OSC, then there exists a basic open set* G *such that* $G \cap K \neq \emptyset$.

Proof. The proof is a modification of [24]. Let W be a basic open set from the OSC, let δ be as in Lemma 2.3(ii), and let C be as in (2.4). We fix an index $\bar{J}$ so that $|W_{\bar{J}}| \leq \delta$ and we let $U = W_{\bar{J}}$. For each J, let

$$\Lambda_n(J) = \left\{I \in \Lambda_{|U_J|} \colon D(U_I, U_J) \leq C^{-n} r_J\right\},$$

and let $\gamma_n = \sup \#\Lambda_n(J)$. Because the previous U_Is are disjoint, each U_J can intersect at most a bounded number (independent of $|U_J|$) of U_I, $I \in \Lambda_n(J)$. We hence have $\gamma_n < \infty$. Furthermore $\{\gamma_n\}$ is decreasing, there exists an $n_0 \geq 1$ such that $\gamma_{n_0} = \gamma_{n_0+1} = \cdots$. We fix this n_0 and we let J_0 be the index such that $\gamma_{n_0+2} = \#\Lambda_{n_0+2}(J_0)$. This implies that $\gamma_{n_0} = \Lambda_{n_0}(J_0)$ also.

For any index I, we use $\mathscr{I}(I)$ to denote the index set $\Lambda_{n_0}(IJ_0)$. Then the maximality of γ_{n_0} implies $\gamma_{n_0} \geq \#\mathscr{I}(I)$. On the other hand observe that for $J \in \Lambda_{n_0}(J_0) = \Lambda_{n_0+2}(J_0)$, by Lemma 2.3(ii) and (2.4),

$$D\left(U_{IJ}, U_{IJ_0}\right) \leq C r_I D\left(U_J, U_{J_0}\right) \leq C r_I C^{-(n_0+2)} r_{J_0} \leq C^{-n_0} r_{IJ_0}.$$

This implies that

$$\mathscr{I}(I) \supseteq \{IJ\colon J \in \Lambda_{n_0}(J_0)\}.$$

Hence $\#\mathscr{I}(I) \geq \#\{IJ\colon J \in \Lambda_{n_0}(J_0)\}$. This, together with $\gamma_{n_0} \geq \#\mathscr{I}(I)$, enables us to see that the preceding inclusion is actually an equality. From this, we conclude that if $I \neq I'$, $r_{I'} \geq r_I$, then

$$D(U_{IJ_0}, U_{I'J_0}) > C^{-n_0} r_{IJ_0}. \tag{2.5}$$

Now define

$$\tilde{U}_{J_0} = \left\{y\colon d(y, U_{J_0}) < \tfrac{1}{2}C^{-n_0-1} r_{J_0}\right\},$$

and let $G = \bigcup_J w_J(\tilde{U}_{J_0})$. Then clearly, $w_j(G) \subseteq G$. We show that $w_i(G) \cap w_j(G) = \varnothing$. Indeed, if $y \in w_i(G) \cap w_j(G)$, there exists $y_1 \in w_{iI}(\tilde{U}_{J_0})$, $y_2 \in w_{jJ}(\tilde{U}_{J_0})$ and

$$d(y_1, y) < \tfrac{1}{2}C^{-n_0-1} r_{iIJ_0}, \qquad d(y_2, y) < \tfrac{1}{2}C^{-n_0-1} r_{jJJ_0}.$$

Without loss of generality assume that $r_{jJJ_0} \geq r_{iIJ_0}$, then we have

$$D(y_1, y_2) \leq C^{-n_0} r_{jJJ_0},$$

which contradicts (2.5). ∎

Proof of Theorem 2.1. Take the basic open set U such that $U \cap K \neq \varnothing$ as in the foregoing lemma. Then $\mu(U) \neq 0$ and Lemma 2.5 implies that μ is concentrated in U. ∎

Proof of Theorem 2.2. Let U be chosen as in the previous text, then the proof of Lemma 2.5 implies that $\mu(\partial U_J) = 0$. Because $K_I \cap K_J \subseteq \overline{U}_I \cap \overline{U}_J$ and $U_I \cap U_J = \varnothing$, we have $K_I \cap K_J \subseteq \partial U_I \cap \partial U_J$ and the result follows. ∎

To conclude this section, we consider now the relation between the self-conformal measure of T_s and the Hausdorff measure on the attractor K. This has also been considered in [16] under some stronger geometric condition on the seed set, the regularity of w_j and the open set condition. We include the following simple proof for completeness.

THEOREM 2.7. *Suppose the conformal family* $\{w_j\}_{j=1}^m$ *satisfies the OSC. Let s be the positive number such that the spectral radius of* T_s *equals* 1 *and let* $\mathscr{H}^s$ *denote the Hausdorff measure. Then* $0 < \mathscr{H}^s(K) < \infty$.

Proof. For any fixed x, let $r_J = |w_J'(x)|$. By Lemma 2.3(ii), we have $|K_J| \leq C r_J$, and hence

$$\sum_{|J|=n} |K_J|^s \leq C \sum_{|J|=n} r_J^s.$$

Note also that

$$\lim_{n\to\infty} \sum_{|J|=n} r_J^s = \lim_{n\to\infty} T_s^n 1(x) = h(x).$$

This implies that $\mathscr{H}^s(K) < \infty$.

Next we prove $\mathscr{H}^s(K) > 0$. We need a measure μ supported by K such that $\mu(B_t) \le Ct^s$ for any ball B_t of radius t, then the mass distribution principle [6] implies $0 < C^{-1} < \mathscr{H}^s(K)$. We take the measure μ satisfying $T_s^* \mu = \mu$, i.e.,

$$\mu = \sum_{j=1}^m \left(|w_j'|^s \mu\right) \circ w_j^{-1}.$$

Let U be a basic open set for K. Then the U_Js, $J \in \Lambda_t$, are disjoint and by Lemma 2.3(ii) there exists a positive number $a > 0$ (independent of t) such that each U_J contains a ball of radius at. This implies that there exists an integer l independent of t such that any ball B_t can intersect at most l of the $\overline{U}_J$ $J \in \Lambda_t$. For a fixed ball B_t, denote this family of J by $\mathscr{F}$. It is easy to show that $\pi^{-1}(B_t) \subseteq \bigcup\{I_J \colon J \in \mathscr{F}\}$ (I_J is the cylinder set with base J) so that $\mu(B_t) \le \sum_{J\in\mathscr{F}} \nu(I_J)$. By (1.2) and Lemma 2.3(i), $\nu(I_J) \le C|w_J'(x)| = Cr_J^s$. It follows that $\mu(B_t) \le C'lt$ and yields the proposition. ∎

LEMMA 2.8. *Let w be conformal and invertible, let E be a Borel subset in the domain of w, and $0 < \mathscr{H}^s(E) < \infty$. Then we have the following formula of change of variable,*

$$\mathscr{H}^s(w(E)) = \int_E |w'(x)|^s \, d\mathscr{H}^s(x).$$

The proof is based on $\mathscr{H}^s(AE) = |\det A|^{s/d} \mathscr{H}^s(E)$ where A is a constant multiple of a rotation, and an elementary approximation technique in change of variable [22].

THEOREM 2.9. *Suppose the conformal family $\{w_j\}_{j=1}^m$ satisfies the OSC. Let $\mu = \mathscr{H}^s|_K$. Then μ is the self-conformal measure for T_s^*, i.e., μ satisfies*

$$\mu = \sum_{j=1}^m \left(|w_j'|^s \mu\right) \circ w_j^{-1}.$$

Proof. Let h be the 1-eigenfunction of T_s. Then

$$h(x) = \sum_{j=1}^m |w_j'(x)|^s h(w_j x). \tag{2.6}$$

It follows that

$$\sum_{j=1}^{m} (h\mu)(K_j) = \sum_{j=1}^{m} \int_{w_j(K)} h(x)\, d\mu(x) = \sum_{j=1}^{m} \int_K h(w_j x)\, d\mu(w_j x)$$
$$= \sum_{j=1}^{m} \int_K |w_j'(x)|^s h(w_j x)\, d\mu(x) = (h\mu)(K).$$

(The third equality is by Lemma 2.8 and the last equality is by (2.6).) This implies that $(h\mu)(K_i \cap K_j) = 0$ for $i \neq j$. Because h is strictly positive, we have $\mu(K_i \cap K_j) = 0$. Now for any Borel subset $E \subseteq K$,

$$\begin{aligned} T_s^*\mu(E) &= \sum_{j=1}^{m} \left(|w_j'|^s \mu\right) \circ w_j^{-1}(E) \\ &= \sum_{j=1}^{m} \int_E \left|w_j'\left(w_j^{-1}(x)\right)\right|^s d\mu\left(w_j^{-1}(x)\right) \\ &= \sum_{j=1}^{m} \int_{w_j^{-1}(E)\cap K} \left|w_j'(y)\right|^s d\mu(y) \\ &= \sum_{j=1}^{m} \mu\left(w_j\left(w_j^{-1}(E) \cap K\right)\right) \\ &= \sum_{j=1}^{m} \mu(E \cap K_j). \end{aligned}$$

Note that $\mu(K_i \cap K_j) = 0$ for $i \neq j$. So the last expression equals $\mu(E)$ and μ is the eigenmeasure. ∎

3. MULTIFRACTAL STRUCTURE

For $0 < t < 1$ we let $\{Q_t(x_i)\}$ denote the family of t-mesh cubes in $\mathbf{R}^d$ with vertices $x_i \in t\mathbf{Z}^d$. Let $Q_t'(x_i)$ be the cube with the same center but each side has length $2t$. Let μ be a bounded positive measure on $\mathbf{R}^d$. For $q \in \mathbf{R}$, $0 < t < 1$, we define the L^q-scaling spectrum,

$$\tau(q) = \liminf_{t\to 0^+} \frac{\log \sum_{i \in \mathscr{F}_t} \mu\left(Q_t'(x_i)\right)^q}{\log t},$$

where $\mathscr{F}_t$ is the family of $Q_t'(x_i)$ such that $Q_t(x_i) \cap \operatorname{supp} \mu \neq \emptyset$. The adjustment of the $Q_t'(x_i)$ in the definition is to guarantee that $Q_t'(x_i)$

intersects a nontrivial portion of supp μ, which is needed for $q < 0$. It is easy to check that $\tau(q)$ is a convex function.

LEMMA 3.1. *Let $(X, \{w_j\}, \{p_j\})$ be a contractive conformal system and let μ be the self-conformal measure. Suppose $\{w_j\}$ satisfies the OSC and let U be a basic open set U supporting μ (as in Theorem 2.1). Let*

$$\Lambda_t = \{J = (j_1 \cdots j_n): n \text{ is the first index such that } |U_J| \le t\}.$$

Then

$$\tau(q) = \liminf_{t\to 0^+} \frac{\log \sum_{J\in\Lambda_t} \mu(U_J)^q}{\log t}.$$

Proof. For each x, there is $\sigma \in \Sigma$ such that $x = \pi(\sigma)$. It follows that $x = \bigcap_{n=1}^{\infty} K_{J_n} = \bigcap_{n=1}^{\infty} \overline{U}_{J_n}$ where $J_n = \sigma|_n$. By the conformal property in Lemma 2.3(ii), it is easy to show that there exist $c_1, c_2 > 0$ (independent of t and x) such that for $J_n \in \Lambda_t$ we have

$$Q_{c_1 t}(x) \subseteq U_{J_n} \subseteq Q_{c_2 t}(x). \tag{3.1}$$

The proof of the lemma for $q \ge 0$ is quite straightforward. We are going to discuss the case $q < 0$. The previous inclusions imply that there exists an integer l (independent of t) such that each U_J $(J \in \Lambda_t)$ intersects at most l of the $Q_t(x_i)$ and vice versa and there exists $a > 0$ such that for $t > 0$ and for $Q'_t(x_i) \in \mathscr{F}_t$, we can find $U_J \subseteq Q'_t(x_i)$ for some $J \in \Lambda_{at}$ (we used the enlarged cube instead of $Q_t(x_i)$ here). This implies that for $Q_t(x_i) \cap \operatorname{supp}\mu \ne \emptyset$, $\mu(Q'_t(x_i))^q \le \mu(U_J)^q$ for $q < 0$ so that

$$\sum_{\mathscr{F}_t} \mu(Q'_t(x_i))^q \le \sum_{J\in\Lambda_{at}} \mu(U_J)^q, \quad \text{for all } t.$$

Similarly we can show that there exists a' such that

$$\sum_{J\in\Lambda_{a't}} \mu(U_J)^q \le \sum_{\mathscr{F}_t} \mu(Q'_t(x_i))^q, \quad \text{for all } t.$$

Hence the lemma follows. ∎

For two given functions φ_1 and φ_2 on Σ and for $q, \tau \in \mathbf{R}$, we define the transfer operator $S_{q,\tau}: C(\Sigma) \to C(\Sigma)$ by

$$S_{q,\tau} f(\sigma) = \sum_{j=1}^{m} e^{q\varphi_1(u_j\sigma) + \tau\varphi_2(u_j\sigma)} f(u_j\sigma).$$

Let $\rho(q,\tau)$ be the spectral radius of $S_{q,\tau}$ and let $P(q,\tau) = \log \rho(q,\tau)$ be the pressure function. The following theorem is well known when φ_1 and

φ_2 are Hölder functions [2, 21, 23]. The same conclusion holds in view of Proposition 1.4.

THEOREM 3.2. *Suppose* $\log \varphi_1$ *and* $\log \varphi_2$ *satisfy the Dini condition. Then*

(i) $P(q,\tau)$ *is convex and real analytic and*

$$\frac{\partial}{\partial q} P(q,\tau) = \int \varphi_1 \, d\tilde{\nu}_{q,\tau}, \qquad \frac{\partial}{\partial \tau} P(q,\tau) = \int \varphi_2 \, d\tilde{\nu}_{q,\tau},$$

where $\tilde{\nu}_{q,\tau}$ *is the Gibbs measure of the system with weight* $e^{q\varphi_1 + \tau\varphi_2}$.

(ii) *If* $\varphi_2 < 0$, *then there exists a real analytic* $\tau\colon \mathbf{R} \to \mathbf{R}$ *such that* $P(q, \tau(q)) = 0$ *and*

$$\tau'(q) = -\frac{\int \varphi_1 \, d\tilde{\nu}_{q,\tau(q)}}{\int \varphi_2 \, d\tilde{\nu}_{q,\tau(q)}}.$$

For the conformal system we define, for $q, \tau \in \mathbf{R}$,

$$T_{q,\tau} f(x) = \sum_{j=1}^{m} \left(\frac{p_j(w_j x)}{\rho} \right)^q |w_j'(x)|^{-\tau} f(w_j x).$$

Note that $T = \rho T_{1,0}$ is the transfer operator in Section 1. If we write $w_j'(x) = w_j'(w_j^{-1}(w_j(x)))$ and we write

$$a_j(x) = \left(\frac{p_j(x)}{\rho} \right)^q \left| w_j'\left(w_j^{-1}(x)\right)\right|^{-\tau},$$

then

$$T_{q,\tau} f(x) = \sum_{j=1}^{m} a_j(w_j x) f(w_j x),$$

and $(X, \{w_j\}, \{a_j\})$ is a contractive system. It is also easy to show that $\log a_j$ satisfies the Dini condition if $\log |w_j'|$ and $\log p_j$ satisfy the same condition. As in the preceding text, we use $\rho(q,\tau)$ to denote the spectral radius of $T_{q,\tau}$ and we use $P(q,\tau) = \log \rho(q,\tau)$ to denote the pressure of the system. It coincides with the one for the corresponding dynamical system (see the proof of Proposition 1.3).

THEOREM 3.3. *Let* $(X, \{w_j\}, \{p_j\})$ *be a contractive conformal system where* $\{w_j\}_{j=1}^m$ *satisfy the OSC and* $\{\log p_j\}_{j=1}^m$ *satisfy the Dini condition. Let* μ *be the self-conformal measure. Then the* L^q*-scaling spectrum* $\tau(q)$ *is the unique solution of the pressure* $P(q, \tau(q)) = 0$.

Proof. That $P(q, \tau) = 0$ means the eigenvalue of $T_{q,\tau}$ is 1, and by Theorem 1.1,

$$\sum_{|J|=n} \left(\frac{p_{w_J}(x)}{\rho^{|J|}} \right)^q |w_J'(x)|^{-\tau} = T_{q,\tau}^n(1) \to h, \text{ uniformly as } n \to \infty.$$

By using the same argument as in Lemma 2.4 we can replace the restriction $|J| = n$ under the summation sign by $J \in \Lambda_t$.

Let U be a basic open set supporting μ. By the Gibbs property (Theorem 1.6), we have for a fixed x,

$$\mu(U_J) = \mu(K_J) \approx \frac{p_{w_J}(x)}{\rho^{|J|}}.$$

Then for $0 < t < 1$,

$$\sum_{J \in \Lambda_t} \mu(U_J)^q \approx \sum_{J \in \Lambda_t} \left(\frac{p_{w_J}(x)}{\rho^{|J|}} \right)^q \approx t^\tau \sum_{J \in \Lambda_t} \left(\frac{p_{w_J}(x)}{\rho^J} \right)^q |w_J'(x)|^{-\tau} \approx t^\tau,$$

as $t \to 0$. This implies that $\tau(q) = \tau$ by Lemma 3.1. ∎

THEOREM 3.4. *Under the same hypothesis as the last theorem and let* $\alpha = \tau'(q)$ *for some* $q \in \mathbf{R}$. *Let*

$$E_\alpha = \left\{ x \in K: \lim_{t \to 0^+} \frac{\log \mu(Q_t(x))}{\log t} = \alpha \right\}.$$

Then $\dim E_\alpha = q \cdot \tau'(q) - \tau(q)$.

Proof. Without loss of generality we assume that the spectral radius of T is equal to 1. Let $K_{J_n}(x)$ denote the sequence of K_{J_n} that converges to x and $J_n = \sigma|_n$ for some sequence of indices σ. By Theorem 1.6 we have $\mu(K_{J_n}(x)) \approx p_{w_{J_n}}(x)$. Let U be the basic open set for $\{w_j\}_{j=1}^m$ as in Theorem 2.1, then $\mu(U_J) = \mu(K_J)$. By using the definition of Λ_t in Lemma 3.1 and applying Lemma 2.3(ii), we have

$$E_\alpha = \left\{ x \in K: \lim_{n \to \infty} \frac{\log \mu\big(K_{J_n}(x)\big)}{\log |w_{J_n}'(x)|} = \alpha \right\}.$$

Let q be chosen so that $\alpha = \tau'(q)$ and μ_q be the self-conformal measure corresponding to $T_{q,\tau(q)}^*$. It follows that μ and μ_q are related by

$$\mu_q\big(K_{J_n}(x)\big) \approx p_{w_{J_n}}(x)^q |w_{J_n}'(x)|^{-\tau(q)} \approx \mu\big(K_{J_n}(x)\big)^q |w_{J_n}'(x)|^{-\tau(q)}.$$

Consequently we have

$$\lim_{n\to\infty} \frac{\log \mu_q\big(K_{J_n}(x)\big)}{\log |w'_{J_n}(x)|} = q \lim_{n\to\infty} \frac{\log \mu\big(K_{J_n}(x)\big)}{\log |w'_{J_n}(x)|} - \tau(q). \tag{3.2}$$

It follows that

$$\lim_{n\to\infty} \frac{\log \mu\big(K_{J_n}(x)\big)}{\log |w'_{J_n}(x)|} = \alpha, \tag{3.3}$$

if and only if

$$\lim_{n\to\infty} \frac{\log \mu_q\big(K_{J_n}(x)\big)}{\log |w'_{J_n}(x)|} = q\alpha - \tau(q). \tag{3.4}$$

By Theorem 1.6(ii) and the expression of $T_{q,\tau}$, we have for μ_q almost all x,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} &\log \mu_q\big(K_{J_n}(x)\big) \\ &= \sum_{j=1}^{m} \int_{K_j} \Big[q \log p_j(y) - \tau(q) \log\Big|w'_j\big(w_j^{-1}(y)\big)\Big|\Big] h_q(y)\, d\mu_q(y).\end{aligned}$$

Furthermore we claim that

$$\lim_{n\to\infty} \frac{1}{n} \log\big|w'_{J_n}(x)\big| = \sum_{j=1}^{m} \int_{K_j} \Big(\log\Big|w'_j\big(w_j^{-1}(y)\big)\Big|\Big) h_q(y)\, d\mu_q(y). \tag{3.5}$$

It follows that for μ_q-almost all x,

$$\begin{aligned}\lim_{n\to\infty} &\frac{\log \mu_q\big(K_{J_n}(x)\big)}{\log |w'_{J_n}(x)|} \\ &= q\left(\frac{\sum_{j=1}^{m} \int_{K_j} \big(\log p_j(y)\big) h_q(y)\, d\mu_q(y)}{\sum_{j=1}^{m} \int_{K_j} \big(\log |w'_j\big(w_j^{-1}(y)\big)|\big) h_q(y)\, d\mu_q(y)}\right) - \tau(q).\end{aligned}$$

By Theorems 3.2(ii) and 3.3 we know that the quotient inside the parenthesis is actually $\tau'(q) = \alpha$. This together with (3.3) and (3.4) imply that μ_q is concentrated in E_α and by the mass distribution principle [6], E_α has Hausdorff dimension $q\tau'(q) - \tau(q)$.

To prove the claim, it will be more convenient to use the notations in the symbolic space as defined in Section 1. Let $\sigma = (j_1, j_2, \dots)$ so that

$\pi(\sigma) = x$. It follows that $x = \lim_{n\to\infty} w_{J_n}(x)$. Furthermore we have $\pi(\theta^i\sigma) = \theta^i\pi(\sigma) = \lim_{n\to\infty} w_{j_{i+1}} \cdots w_{j_n}(x)$. We define

$$f(\sigma) = f(j_1, \theta(\sigma)) = \log |w'_{j_1}(\pi \circ \theta(\sigma))|.$$

Then

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\log |w'_{J_n}(x)| &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log |w'_{j_i}\big(w_{j_{i+1}} \cdots w_{j_n}(x)\big)| \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log |w'_{j_i}\big(\pi \circ \theta^i(\sigma)\big)| \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log |f\big(\theta^i(\sigma)\big)|.\end{aligned}$$

For the second equality, we used the Dini condition on w_js. Let $\tilde{\nu}_q$ be the invariant measure on Σ corresponding to $h_q \mu_q$ on K. Then by the Ergodic theorem, for $\tilde{\nu}_q$ almost all σ the preceding limit equals

$$\begin{aligned}\int_{\Sigma} f(\omega)\, d\tilde{\nu}_q(\omega) &= \sum_{j=1}^{m} \int_{\Sigma_j} f(j, \theta(\omega)) h_q(\pi\omega)\, d\nu_q(\omega) \\ &= \sum_{j=1}^{m} \int_{\Sigma_j} \big(\log \big| w'_j(\pi\theta(\omega))\big|\big) h_q(\pi\omega)\, d\nu_q(\omega) \\ &= \sum_{j=1}^{m} \int_{\Sigma_j} \big(\log \big| w'_j\big(\pi u_j^{-1}(\omega)\big)\big|\big) h_q\big(\pi u_j u_j^{-1}(\omega)\big)\, d\nu_q(\omega) \\ &= \sum_{j=1}^{m} \int_{\Sigma} \big(\log \big| w'_j(\pi\omega)\big|\big) h_q\big(\pi u_j(\omega)\big)\, d\nu_q(j, \omega) \\ &= \sum_{j=1}^{m} \int_{\Sigma} \big(\log \big| w'_j(\pi\omega)\big|\big) h_q(\omega_j \pi\omega)\, d\nu_q(j, \omega) \\ &= \sum_{j=1}^{m} \int_{K} \big(\log \big| w'_j(y)\big|\big) h_q(w_j y)\, d\mu_q(j, y) \\ &= \sum_{j=1}^{m} \int_{K_j} \big(\log \big| w'_j\big(w_j^{-1}x\big)\big|\big) h_q(x)\, d\mu_q(x),\end{aligned}$$

where $\nu_q(j, \cdot)$ and $\mu_q(j, \cdot)$ are defined as $\nu(j, \cdot)$ and $\mu(j, \cdot)$ in the proof of Theorem 1.6, and the calculation is also the same as there. This implies (3.5) and completes the proof. ∎

We define the Hausdorff dimension and the entropy dimension of μ as

$$\dim_h \mu = \inf\{\dim_h(E): \mu(E) = 1\},$$

and

$$\dim_e \mu = \liminf_{t \to 0^+} \frac{\log \sum_{i \in \mathscr{F}_t} \mu(Q_t(x_i)) \log \mu(Q_t(x_i))}{\log t},$$

where $\mathscr{F}_t$ is the family of $Q_t(x_i)$ such that $Q_t(x_i) \cap \operatorname{supp} \mu \neq \phi$. See [7] for other related notions. It is known that if $\tau'(1)$ exists, then $\tau'(1) = \dim_h \mu = \dim_e \mu$ [15].

COROLLARY 3.5. *Under the previous assumption, then*

$$\dim_h \mu = \dim_e \mu = \tau'(1) = \frac{\sum_{j=1}^m \int_{K_j} (\log p_j(x)) h(x)\, d\mu(x)}{\sum_{j=1}^m \int_{K_j} (\log |w_j'(w_j^{-1}(x))| h(x)\, d\mu(x)}.$$

In particular we see that when $p_j(x) = p_j$ are constants and $\sum p_i = 1$, then $h = 1$ and $\mu = \sum_{j=1}^m p_j \mu \circ w_j^{-1}$ the preceding quantity can easily be reduced to

$$\frac{\sum_{j=1}^m p_j \log p_j}{\sum_{j=1}^m p_j \int_K \log |w_j'(x)|\, d\mu(x)}.$$

It coincides with the result obtained by Strichartz [25] using a family of vector-valued self-similar measures to approximate the self-conformal measures.

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