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L^q-spectrum of the Bernoulli convolution associated with the golden ratio

by

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Abstract. Based on a set of higher order self-similar identities for the Bernoulli convolution measure for $(\sqrt{5}-1)/2$ given by Strichartz *et al.*, we derive a formula for the L^q-spectrum, $q > 0$, of the measure. This formula is the first obtained in the case where the open set condition does not hold.

1. Introduction. Let μ be a positive bounded regular Borel measure on $\mathbb{R}^d$ with compact support. For $h > 0$ and $q > 0$, we define the L^q-*(moment) spectrum* of μ by

$$\tau(q) = \varliminf_{h\to 0^+} \frac{\ln \sum_i \mu(Q_i(h))^q}{\ln h}, \tag{1.1}$$

where $\{Q_i(h)\}_i$ is the family of h-mesh cubes

$$[n_1 h, (n_1+1)h) \times \ldots \times [n_d h, (n_d+1)h), \quad (n_1, \ldots, n_d) \in \mathbb{Z}^d.$$

We also define the *(lower)* L^q-*dimension* of μ by

$$\underline{\dim}_q(\mu) = \tau(q)/(q-1), \quad q > 1.$$

These notions were first used by Rényi [Ré] to extend the entropy dimension (corresponding to $q = 1$). Some variants of these definitions and the basic properties of $\tau(q)$ can be found in [LN1], [St]. We prefer to use $\varliminf$ rather than $\varlimsup$ because the $\tau(q)$ defined by using $\varliminf$ is concave.

Recently there are a large number of papers in the mathematics and physics literature investigating the relationship of the L^q-spectrum and the local dimension of the measures that arise from dynamical systems (the multifractal formalism) (e.g., Frisch and Parisi [FP], Halsey *et al.* [H], Collet *et al.* [CLP], Lopes [Lo], Rand [R], Cawley and Mauldin [CM], Edgar and

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Mauldin [EM], Lau and Ngai [LN1], Olsen [O], Riedi [Ri], Daubechies and Lagarias [DL]). The multifractal formalism asserts, heuristically, that the Legendre transformation of $\tau(q)$, i.e., $\tau^*(\alpha) := \inf\{q\alpha - \tau(q) : q \in \mathbb{R}\}$, is equal to the Hausdorff dimension of the set

$$K(\alpha) := \left\{x \in \mathrm{supp}(\mu) : \lim_{h\to 0^+} \frac{\ln \mu(B_h(x))}{\ln h} = \alpha\right\},$$

where $\mathrm{supp}(\mu)$ denotes the support of μ, $B_h(x)$ is the closed h-ball centered at x, and the quantity

$$\lim_{h\to 0^+} \frac{\ln \mu(B_h(x))}{\ln h}$$

is known as the *local dimension* of μ at x. The Hausdorff dimension of $K(\alpha)$, as a function of α, is the well-known *dimension spectrum*, and if the multifractal formalism holds, then it can be obtained indirectly by calculating $\tau(q)$.

For a rigorous verification of the multifractal formalism and an explicit calculation of the L^q-spectrum, it is customary to restrict to the class of self-similar measures (or its variants), i.e., probability measures μ satisfying

$$\mu = \sum_{j=1}^{m} a_j \mu \circ S_j^{-1}, \tag{1.2}$$

where $\{S_j\}_{j=1}^m$ are contractive similitudes and $\{a_j\}_{j=1}^m$ are probability weights [Hut]. A further restriction is that the similitudes must satisfy a certain separation condition, called the *open set condition* ([Hut], [F]). The formula for $\tau(q)$ is then given by

$$\sum_{j=1}^{m} a_j^q \varrho_j^{-\tau(q)} = 1,$$

where ϱ_j is the contraction ratio of S_j ([CM], [EM], [LW], [St]).

Very little is known when the similitudes do not satisfy such a condition: the simplest case is when $m = 2$ and

$$S_1(x) = \varrho x, \qquad S_2(x) = \varrho x + (1-\varrho), \tag{1.3}$$

where $1/2 < \varrho < 1$ and the weight on each map is $1/2$. The corresponding self-similar measure μ equals the distribution measure of the random variable $(1-\varrho)\sum_{n=0}^{\infty} \varrho^n \varepsilon_n$ where $\{\varepsilon_n\}_{n=0}^{\infty}$ are i.i.d. random variables taking values 0 and 1 with probability $1/2$. The measure in such context has been studied for a long time and is called the *infinitely convolved Bernoulli measure* (ICBM).

In contrast to the case where $0 < \varrho < 1/2$, which gives a Cantor type measure, the case of $1/2 < \varrho < 1$ is rather complicated due to the overlapping of the two sets $S_1[0,1]$ and $S_2[0,1]$, where $[0,1]$ is the support of μ. There is still no satisfactory condition to determine whether a measure μ in this class is singular or absolutely continuous. Recently Solomyak [So] proved the important result that for almost all $1/2 < \varrho < 1$, the corresponding μ is absolutely continuous. However, his theorem does not tell us which ϱ gives an absolutely continuous or a singular measure. So far the only known condition for such μ to be singular is due to Erdős ([E], [S]): ϱ^{-1} is a P.V. number. (Recall that $\beta > 1$ is a *Pisot–Vijayaraghavan* (*P.V.*) *number* if it is an algebraic integer and all its conjugates have moduli less than 1. The most celebrated P.V. number is the golden ratio $\beta = (\sqrt{5}+1)/2$.)

Recently the authors introduced a *weak separation condition* on the generating similitudes $\{S_j\}_{j=1}^m$ [LN1]. This new condition covers the previous open set condition and is satisfied by the maps in (1.3) when ϱ^{-1} is a P.V. number. Under this condition the multifractal formalism was proved to hold. This makes the computation of $\tau(q)$ an important and natural problem. However, the explicit calculation of the L^q-spectrum in general is still an unsettled question. It is hence desirable to obtain a complete understanding of such spectrum for the ICBM μ defined by the P.V. numbers.

Historically the interest has been in the entropy dimension (i.e., $q = 1$, see Section 4) of the measure (Garsia [G]). This dimension has been reexamined more recently by Alexander and Yorke [AY], Alexander and Zagier [AZ], Lalley [La], Ledrappier and Porzio [LP], Przytycki and Urbański [PU]. On the other hand, the L^2-dimension has been calculated in [L1], [L2], and the method used there has been extended to study the L^q-dimension when q is a positive integer [LN2]. Also the L^∞-dimension (see Section 4) of the measure studied in this paper has been obtained by Hu [Hu].

In this paper our goal is to obtain an exact formula for the L^q-spectrum, $q > 0$, for the ICBM when $\varrho = (\sqrt{5}-1)/2$. The basic idea is to use a device introduced by Strichartz *et al.* [STZ] to decompose the overlapping of $S_1[0,1]$ and $S_2[0,1]$ into nonoverlapping sets by compositions of S_1 and S_2. Let

$$\begin{aligned} T_0x &= S_1S_1x = \varrho^2 x,\\ T_1x &= S_1S_2S_2x = S_2S_1S_1x = \varrho^3 x + \varrho^2,\\ T_2x &= S_2S_2x = \varrho^2 x + \varrho. \end{aligned} \tag{1.4}$$

Then $T_0[0,1] = [0,\varrho^2]$, $T_1[0,1] = [\varrho^2,\varrho]$, $T_2[0,1] = [\varrho,1]$ are three intervals with disjoint interiors. In terms of these maps, the self-similar identity (1.2) (more precisely (2.1)) yields three sets of second-order identities: For $A \subseteq [0,1]$,

$$\begin{bmatrix} \mu(T_0T_iA)\\ \mu(T_1T_iA)\\ \mu(T_2T_iA)\end{bmatrix} = M_i \begin{bmatrix} \mu(T_0A)\\ \mu(T_1A)\\ \mu(T_2A)\end{bmatrix}, \qquad i = 0,1,2, \tag{1.5}$$

where

$$M_0 = \begin{bmatrix} \frac14 & 0 & 0 \\ \frac18 & \frac14 & 0 \\ 0 & \frac12 & 0 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0 & \frac14 & 0 \\ 0 & \frac14 & 0 \\ 0 & \frac14 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & \frac12 & 0 \\ 0 & \frac14 & \frac18 \\ 0 & 0 & \frac14 \end{bmatrix}.$$

For any integer $k \ge 0$, let $J = (j_1, \dots, j_k)$, $j_i = 0$ or 2, and let

$$c_J = \frac14 [0 \quad 1 \quad 0]\, M_J \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \tag{1.6}$$

where $M_J = M_{j_1} \dots M_{j_k}$. Define

$$\begin{aligned} F(q,\alpha) &= \sum_{k=0}^{\infty} \varrho^{-(2k+3)\alpha} \Big(\sum_{|J|=k} c_J^q \Big), \\ D &= \{(q,\alpha) : q > 0,\ F(q,\alpha) < \infty\}, \end{aligned} \tag{1.7}$$

i.e., D is the domain of convergence of the series. Our main theorem concerning the L^q-spectrum of μ is

THEOREM A. *If $q > 0$, then $\tau(q)$ is equal to the unique α satisfying $F(q,\alpha) = 1$. Moreover, the domain of convergence D is open and τ is differentiable at q.*

The differentiability of $\tau(q)$ implies that its Legendre transformation τ^* is strictly concave. Hence we can apply Theorem 6.6 of [LN1] to obtain the following

COROLLARY. *For the above μ, the multifractal formalism holds for $q > 0$, i.e., $\tau^*(\eta)$ is equal to the Hausdorff dimension of $K(\eta)$ where $\eta = \tau'(q)$.*

The proof of Theorem A depends on an accurate estimation of

$$\Phi^{(\alpha)}(h) = \frac{1}{h^{1+\alpha}} \int_0^1 \mu(B_h(T_1 x))^q \, dx,$$

where $B_h(x)$ is the ball of radius h centered at x. This term is equivalent to $h^{-\alpha} \sum_i \mu(Q_i(h))^q$ in the definition of $\tau(q)$ (see [St], [L1]). By using (1.2) and (1.5) we establish a functional equation for $\Phi^{(\alpha)}(h)$ in terms of the c_J's (see (3.8)), which is of the form of a renewal equation [Fe]. We then apply the renewal theorem to show that if $F(q,\alpha) = 1$, then $0 < \underline{\lim}_{h\to 0^+} \Phi^{(\alpha)}(h) < \infty$. This is used to show that $\tau(q) = \alpha$. A similar method had been used in [L1] and [LW] to calculate the L^2-dimension of some self-similar measures.

By changing bases we can simplify (1.6) by replacing M_0, M_2 with

$$P_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

and show that

$$c_J = \frac{1}{2 \cdot 4^{k+1}} [1 \quad 1]\, P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where $P_J = P_{j_1} \dots P_{j_k}$, $j_k = 0$ or 2 (Proposition 2.3). For the case where q is a nonnegative integer, we can prove the following result, which simplifies the calculation of $\tau(q)$ for such q.

THEOREM B. *For q a nonnegative integer, the equation $F(q,\alpha) = 1$ can be reduced to a polynomial equation $P(z) = 0$ (with $z = 2^q \varrho^\alpha$). In this case, $\tau(q) = \ln(z/2^q)/\ln \varrho$, where z is the largest positive root of P.*

Theorem B corresponds to Theorem 4.1 where $P(z) = 0$ is expressed as a rational function equation instead. We remark that for nonnegative integers q, the values for $\tau(q)$ obtained by solving the polynomial equation in Theorem B coincide with those obtained in [L1], [L2] and [LN2] by using different methods. Also we apply Theorem A and conclude that

THEOREM C. *The Hausdorff dimension of μ is the same as its entropy dimension and equals*

$$\tau'(1) = \frac{1}{9 \ln \varrho} \sum_{k=0}^{\infty} \sum_{|J|=k} c_J \ln c_J \quad (\approx 0.9957).$$

THEOREM D.

$$\underline{\dim}_\infty(\mu) = \left| \frac{\ln 2}{\ln \varrho} \right| - \frac12 \quad (\approx 0.9404).$$

The above approximate value of the entropy dimension agrees with those computed in [AY], [AZ] and [La], and the L^∞-dimension of μ coincides with the result obtained in [Hu].

We organize this paper as follows. In Section 2, we give a detailed study of the second-order identities for μ associated with T_0, T_1 and T_2, and obtain initial estimations for the c_J. We also investigate the domain of convergence D as defined in (1.7) and show that it is open. This is essential in the proof of Theorem A. In Section 3, we derive the basic functional equation (3.8) and prove Theorem A by using the renewal theorem. The proof depends on a technical lemma on estimating the error term (Lemma 3.3). For clarity we postpone its proof to Section 5. In Section 4, we prove Theorem B (i.e., Theorem 4.1). We also derive formulas for the L^q-dimension for q a positive integer, the Hausdorff and entropy dimensions, and the L^∞-dimension of μ.

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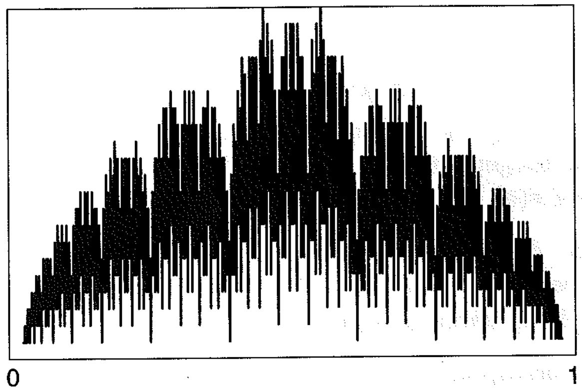

Fig. 1. Graph showing the approximate local density for the measure associated with the golden ratio $\varrho = (\sqrt{5}-1)/2$ over the support of the measure $[0,1]$

to thank Professor R. Strichartz for arranging a visit to Cornell University where part of this research was carried out.

2. Second-order identities. For $\varrho = (\sqrt{5}-1)/2$, we have $\varrho^2 + \varrho = 1$ and the self-similar identity (1.2) of the ICBM μ becomes

$$\mu(E) = \tfrac{1}{2}\mu(\varrho^{-1}E) + \tfrac{1}{2}\mu(\varrho^{-1}E - \varrho). \tag{2.1}$$

The measure is supported by $[0,1]$ and is symmetric about $1/2$. For any integer $k \geq 0$, we use $J = (j_1, \ldots, j_k)$, $j_i = 0$ or 2, to denote a multi-index and let $|J|$ denote the length of J. (By convention, $J = \emptyset$ if $|J| = 0$.) We also write $T_J = T_{j_1} \circ \ldots \circ T_{j_k}$.

We remark that by iterating the T_i's, it is easy to see that the subintervals in the collection $\{T_1 T_J T_1[0,1] : |J| \geq 0,\ j_i = 0 \text{ or } 2\}$ are disjoint and fill up the interval $T_1[0,1]$. It will become clear in Section 3 that this explains why only those J with $j_i = 0$ or 2 appear in formulas (1.6) and (1.7).

The following proposition is most useful in deriving the formula of the L^q-spectrum of μ.

PROPOSITION 2.1. *Let c_J be defined as in* (1.6).

(i) *If* $A \subseteq [0,1]$, *then* $\mu(T_1 T_J T_1 A) = c_J \mu(T_1 A)$;
(ii) *if* $A \subseteq [-1,2]$, *then* $c_J \mu(T_1 A) \leq \mu(T_1 T_J T_1 A) \leq 2 c_J \mu(T_1 A)$.

Proof. (i) The assertion follows by repeated applications of (1.5). Let $J = (j_1, J')$. Then

$$\mu(T_1 T_J T_1 A) = [0\ \ 1\ \ 0]\, M_{j_1} \begin{bmatrix} \mu(T_0 T_{J'} T_1 A) \\ \mu(T_1 T_{J'} T_1 A) \\ \mu(T_2 T_{J'} T_1 A) \end{bmatrix} = \ldots$$

$$= [0\ \ 1\ \ 0]\, M_J \begin{bmatrix} \mu(T_0 T_1 A) \\ \mu(T_1 T_1 A) \\ \mu(T_2 T_1 A) \end{bmatrix}$$

$$= [0\ \ 1\ \ 0]\, M_J M_1 \begin{bmatrix} \mu(T_0 A) \\ \mu(T_1 A) \\ \mu(T_2 A) \end{bmatrix}$$

$$= [0\ \ 1\ \ 0]\, M_J \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} \mu(T_1 A) = c_J \mu(T_1 A).$$

(ii) Let $B = A \cap [-1,0]$. By applying (2.1) twice, we have

$$\begin{bmatrix} \mu(T_0 T_1 B) \\ \mu(T_1 T_1 B) \\ \mu(T_2 T_1 B) \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \mu(T_1 B) \\ \mu(T_1 B + \varrho) \end{bmatrix}.$$

(Note that this identity is different from the expression in (1.5), which only holds for subsets in $[0,1]$.) Another application of (2.1) yields

$$\mu(T_1 B) = \tfrac{1}{4}\mu(\varrho B + 1) + \tfrac{1}{4}\mu(\varrho B + \varrho^2) \quad \text{and} \quad \mu(T_1 B + \varrho) = \tfrac{1}{4}\mu(\varrho B + 1)$$

so that $\mu(T_1 B + \varrho) \leq \mu(T_1 B)$. Now we can repeat the same iteration as in (i) and obtain

$$\mu(T_1 T_J T_1 B) = [0\ \ 1\ \ 0]\, M_J \begin{bmatrix} \mu(T_0 T_1 B) \\ \mu(T_1 T_1 B) \\ \mu(T_2 T_1 B) \end{bmatrix}$$

$$\leq [0\ \ 1\ \ 0]\, M_J \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \mu(T_1 B) \\ \mu(T_1 B) \end{bmatrix}$$

$$\leq [0\ \ 1\ \ 0]\, M_J \begin{bmatrix} \frac{1}{4} \\ \frac{3}{8} \\ \frac{1}{2} \end{bmatrix} \mu(T_1 B) \leq 2 c_J \mu(T_1 B).$$

The lower bound estimate can be more easily obtained by using

$$\begin{bmatrix} \mu(T_0 T_1 B) \\ \mu(T_1 T_1 B) \\ \mu(T_2 T_1 B) \end{bmatrix} \geq \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} \mu(T_1 B).$$

The same argument holds for $A \cap [1,2]$. Now the result follows by summing the three components of A in $[-1,2]$ and using (i). ∎

COROLLARY 2.2. *If* $A \subseteq [0,1]$ *and* $J = (j_1, \ldots, j_k)$ *with* $j_i = 0$ *or* 2, *then*

$$\mu(T_1 T_J T_j A) \le 2c_J, \qquad j = 0 \text{ or } 2.$$

Proof. By using (1.5) and a similar derivation as in the proof of Proposition 2.1(i) we have

$$\mu(T_1 T_J T_0 A) = [0 \quad 1 \quad 0]\, M_J M_0 \begin{bmatrix} \mu(T_0 A) \\ \mu(T_1 A) \\ \mu(T_2 A) \end{bmatrix} \le [0 \quad 1 \quad 0]\, M_J \begin{bmatrix} \frac14 \\ \frac38 \\ \frac12 \end{bmatrix} \le 2c_J.$$

The same argument holds for $\mu(T_1 T_J T_2 A)$. ∎

We will now simplify the c_J's by replacing the 3×3 matrices M_0 and M_2 with the 2×2 matrices

$$P_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

PROPOSITION 2.3. *Let* $k \ge 0$ *and* $J = (j_1, \ldots, j_k)$ *with* $j_i = 0$ *or* 2. *Then*

$$c_J = \frac{1}{2 \cdot 4^{k+1}} [1 \quad 1]\, P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Proof. We will prove the proposition when the last index i_k of J is 0, i.e., $M_{j_k} = M_0$ and $P_{j_k} = P_0$. By using the matrices

$$S = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \qquad S^{-1} = \frac12 \begin{bmatrix} -1 & 2 & 0 \\ 1 & 0 & 0 \\ 2 & -4 & 2 \end{bmatrix},$$

we define

$$Q_0 := S^{-1} M_0 S = \frac14 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad Q_2 := S^{-1} M_2 S = \frac14 \begin{bmatrix} 1 & 0 & \frac12 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence

$$\begin{aligned} c_J &= \frac14 [0 \quad 1 \quad 0]\, M_J \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac14 [0 \quad 1 \quad 0]\, (S Q_{j_1} \cdots Q_{j_k} S^{-1}) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{4^{k+1}} [0 \quad 1 \quad 0]\, S \begin{bmatrix} P_{j_1} \cdots P_{j_k} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} S^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad (\text{use } Q_{j_k} = Q_0) \\ &= \frac{1}{2 \cdot 4^{k+1}} [1 \quad 1]\, P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad (\text{by a direct calculation}). \end{aligned}$$

In the case where the last index i_k of J is 2, i.e., $M_{j_k} = M_2$ and $P_{j_k} = P_2$, we use

$$T = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix}, \qquad T^{-1} = \frac12 \begin{bmatrix} 2 & -4 & 2 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

instead of S and S^{-1} and the proof follows as above. ∎

PROPOSITION 2.4. *Let* $J = (j_1, \ldots, j_k)$, $j_i = 0$ *or* 2. *Then*

(i) $\displaystyle\sum_{k=0}^{\infty} \sum_{|J|=k} c_J = 1$, *and*

(ii) $\displaystyle\max\{c_J : |J| = k\} \le \frac{1}{4(4\varrho)^k} \le \frac{1}{2^{k+2}}$

for all $k \ge 0$. *Moreover,* $\max\{c_J : |J| = k\}$ *has the same order as* $1/(4\varrho)^k$ $(\approx (0.4045)^k)$ *as* $k \to \infty$.

Proof. (i) We note that by Proposition 2.3,

$$\sum_{|J|=k} c_J = \sum_{J=k} \frac{1}{2 \cdot 4^{k+1}} [1 \quad 1]\, P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac18 [1 \quad 1] \left(\frac14 (P_0 + P_2) \right)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

A direct calculation yields

$$\left(I - \frac14 (P_0 + P_2) \right)^{-1} = \frac43 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

so that

$$\sum_{k=0}^{\infty} \sum_{|J|=k} c_J = \frac18 [1 \quad 1] \left(I - \frac14 (P_0 + P_2) \right)^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1.$$

We remark that the above result can also be derived from Proposition 2.1(i) with $A = [0,1]$, by using the fact that the subintervals $T_1 T_J T_1 [0,1]$ are disjoint and fill up $T_1[0,1]$.

(ii) We observe that the products $P_0 P_0$, $P_0 P_2$, $P_2 P_0$, $P_2 P_2$ are

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

respectively. It follows by induction that for even k, the alternating product $(P_0 P_2)^{k/2}$ (resp. $(P_2 P_0)^{k/2}$) maximizes simultaneously $[1 \quad 1]\, P_J \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ and the first (resp. second) column sum of P_J over all J with $|J| = k$. For odd k, this holds for the alternating products $(P_2 P_0)^{(k-1)/2} P_2$ and $(P_0 P_2)^{(k-1)/2} P_0$ respectively. Hence the maximum of $[1 \quad 1]\, P_J \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ for $|J| = k$ is attained by multiplying P_0 and P_2 alternately. Both $P_0 P_2$ and $P_2 P_0$ have maximum eigenvalue $(\sqrt{5}+3)/2$, which equals ϱ^{-2}. This implies that $\max\{c_J : |J| = k\}$ is of the same order as $1/(4\varrho)^k$ as $k \to \infty$.

To prove the stated upper bound estimate for c_J we diagonalize P_0P_2 (and similarly P_2P_0) by using the matrix $S := \begin{bmatrix} 1 & -\varrho \\ \varrho & 1 \end{bmatrix}$ whose columns are eigenvectors of P_0P_2 associated with the eigenvalues ϱ^{-2} and ϱ^2 respectively. If $k = 2l$, $l \geq 0$, then

$$[1 \quad 1](P_0P_2)^l \begin{bmatrix} 1 \\ 1 \end{bmatrix} = [1 \quad 1] S \begin{bmatrix} \varrho^{-2l} & 0 \\ 0 & \varrho^{2l} \end{bmatrix} S^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$= \frac{(2+\varrho+\varrho^{4l+4})}{1+\varrho^2}\varrho^{-2l} \leq 2\varrho^{-2l}.$$

Similarly, if $k = 2l+1$, $l \geq 0$, then

$$[1 \quad 1](P_0P_2)^l P_0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{(2+\varrho-\varrho^{4l+6})}{1+\varrho^2}\varrho^{-(2l+1)} < 2\varrho^{-(2l+1)}.$$

In both cases we see that $c_J \leq 1/(4(4\varrho)^k)$. ∎

We now examine the domain of convergence of D for the series $F(q,\alpha)$ in (1.7).

PROPOSITION 2.5. *Let $F(q,\alpha)$ and D be given by* (1.7). *Then*

(i) *D is an open convex set;*

(ii) *if $(q,\widetilde{\alpha}) \in \partial D$, the boundary of D, then $\widetilde{\alpha} = \widetilde{\alpha}_q$ is an increasing concave function of q;*

(iii) *there exists a unique α such that $(q,\alpha) \in D$ and $F(q,\alpha) = 1$. Moreover, $\alpha = \alpha(q)$ is a differentiable, strictly increasing concave function.*

Proof. We first prove the following claim: Let $\{a_k\}$ be a nonnegative sequence of real numbers such that $a_{m+k} \leq a_m a_k$ for all m, k. Suppose $\overline{\lim}_{k\to\infty} a_k < 1$. Then there exist some $0 < r < 1$ and a constant $C > 0$ such that $a_k \leq Cr^k$ for all k.

To prove the claim we observe that by the assumption, there exists k_0 such that $a_{k_0} < 1$. Hence for $k > k_0$,

$$a_k \leq a_{k_0} \cdot a_{k-k_0} \leq a_{k_0}^2 \cdot a_{k-2k_0} \leq \ldots \leq C \cdot a_{k_0}^{k/k_0} = Cr^k,$$

where $r = a_{k_0}^{1/k_0} < 1$.

Now assume $q > 0$ and let

$$s_k := \sum_{|J|=k} \left([1 \quad 1] P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)^q.$$

We first observe that $s_{m+k} \leq s_m s_k$. In fact,

$$s_m s_k = \sum_{|J|=m} \left([1 \quad 1] P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)^q \cdot \sum_{|J'|=k} \left([1 \quad 1] P_{J'} \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)^q$$
$$= \sum_{|J|=m}\sum_{|J'|=k} \left([1 \quad 1] P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1 \quad 1] P_{J'} \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)^q$$
$$\geq \sum_{|J|=m}\sum_{|J'|=k} \left([1 \quad 1] P_J P_{J'} \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)^q = s_{m+k}.$$

Write $s_k = a_k R^k$, where $R = \overline{\lim}_{k\to\infty} \sqrt[k]{s_k}$. Then $s_{m+k} \leq s_m s_k$ implies that $a_{m+k} \leq a_m a_k$. By the claim just proved and the definition of $\{a_k\}$, we have $\overline{\lim}_{k\to\infty} a_k \geq 1$. Consequently for $(q,\widetilde{\alpha}) \in \partial D$, $F(q,\alpha)$ tends to infinity as α tends to $\widetilde{\alpha}$ and

$$F(q,\widetilde{\alpha}) = \sum_{k=0}^{\infty} \varrho^{-(2k+3)\widetilde{\alpha}} \Big(\sum_{|J|=k} c_J^q\Big) = \varrho^{-3\widetilde{\alpha}} \sum_{k=0}^{\infty} a_k = \infty.$$

Hence D is open and the solution (q,α) for $F(q,\alpha) = 1$ satisfies $(q,\alpha) \in D$. Hölder's inequality applied to $F(q,\alpha)$ implies that D is convex and the Weierstrass M-test shows that for $(q_0,\alpha_0) \in D$, $F(q,\alpha)$ converges uniformly on the region $\{(q,\alpha) : q \geq q_0, \ \alpha \leq \alpha_0\}$. Hence $\widetilde{\alpha}_q$ is an increasing concave function of q. Lastly, it is obvious that $\alpha(q)$, defined by $F(q,\alpha(q)) = 1$, is strictly increasing. The differentiability of $\alpha(q)$ follows by applying the implicit function theorem, and the concavity of $\alpha(q)$ can be verified by simple applications of Hölder's inequality. ∎

We will show that $\alpha(q) = \tau(q)$. See Figure 2 for the graphs of $\widetilde{\alpha}_q$ and $\tau(q)$.

3. Formula for the L^q-spectrum. For $q > 0$ the L^q-spectrum $\tau(q)$ in (1.1) can be expressed as

$$\tau(q) = \inf\Big\{\alpha : \underline{\lim}_{h\to 0+} \frac{1}{h^\alpha} \sum_i \mu(Q_i(h))^q > 0\Big\} \tag{3.1}$$
$$= \inf\Big\{\alpha : \underline{\lim}_{h\to 0+} \frac{1}{h^{1+\alpha}} \int_0^1 \mu(B_h(x))^q\,dx > 0\Big\}.$$

In our approach, we find it more convenient to use $h^{-1-\alpha}\int_0^1 \mu(B_h(x))^q\,dx$ instead of $h^{-\alpha}\sum_i \mu(Q_i(h))^q$. (Note that these two expressions dominate each other by positive constants [L1], [St].) In the proof it is necessary to convert the above expression into an integral in terms of the map T_1. Thus we let

$$(3.2)\qquad \varphi(h)=\int_0^1 \mu(B_h(T_1x))^q\,dx \quad\text{and}\quad \Phi^{(\alpha)}(h)=\frac{1}{h^{1+\alpha}}\varphi(h).$$

PROPOSITION 3.1. *If* $0<\overline{\lim}_{h\to 0^+}\Phi^{(\alpha)}(h)<\infty$, *then* $\tau(q)=\alpha$.

Proof. To find $\tau(q)$ it suffices to look for α so that

$$(3.3)\qquad 0<\overline{\lim_{h\to 0^+}}\,\frac{1}{h^{1+\alpha}}\int_0^1 \mu(B_h(x))^q\,dx<\infty.$$

We can further replace $\mu(B_h(x))$ in the integrand by $\mu(B_h(T_1x))$ as follows: Observe that by a change of variables,

$$\varphi(h)=\int_0^1 \mu(B_h(\varrho^3x+\varrho^2))^q\,dx=\varrho^{-3}\int_{\varrho^2}^{\varrho}\mu(B_h(x))^q\,dx.$$

By (2.1) and another change of variables, we have

$$\int_{\varrho^2}^{\varrho}\mu(B_h(x))^q\,dx=\int_{\varrho^2}^{\varrho}\Big(\frac12\mu(B_{h/\varrho}(\varrho^{-1}x))+\frac12\mu(B_{h/\varrho}(\varrho^{-1}x-\varrho))\Big)^q\,dx$$
$$\geq\frac{1}{2^q}\int_{\varrho^2}^{\varrho}\mu(B_{h/\varrho}(\varrho^{-1}x))^q\,dx=\frac{\varrho}{2^q}\int_{\varrho}^{1}\mu(B_{h/\varrho}(x))^q\,dx.$$

Similarly, we use the part $\mu(B_{h/\varrho}(\varrho^{-1}x-\varrho))$ in the above expression to obtain

$$\int_{\varrho^2}^{\varrho}\mu(B_h(x))^q\,dx\geq\frac{\varrho}{2^q}\int_0^{\varrho^2}\mu(B_{h/\varrho}(x))^q\,dx.$$

If we write

$$\int_0^1\mu(B_h(x))^q\,dx=\Big(\int_0^{\varrho^2}+\int_{\varrho^2}^{\varrho}+\int_{\varrho}^{1}\Big)\mu(B_h(x))^q\,dx,$$

then the three terms are bounded above and below by some constant multiples of $\varphi(h)$. This implies (3.3) and hence the proposition follows. ∎

Next we will derive a functional equation for $\Phi^{(\alpha)}(h)$, i.e., the equation in (3.8). Note that

$$(3.4)\qquad \varphi(h)=\Big(\int_{T_0[0,1]}+\int_{T_1[0,1]}+\int_{T_2[0,1]}\Big)\mu(B_h(T_1x))^q\,dx$$
$$=\varrho^2\int_0^1\mu(B_h(T_1T_0x))^q\,dx+\varrho^3\int_0^1\mu(B_h(T_1T_1x))^q\,dx$$

$$+\varrho^2\int_0^1\mu(B_h(T_1T_2x))^q\,dx.$$

We keep the middle term T_1T_1 and repeat the above process on the first and third terms. There are six terms $T_1T_jT_k$ where $j=0,2$ and $k=0,1,2$. We again keep the two terms $T_1T_jT_1$ and iterate the other four terms. By repeating this process for $N+1$ steps, we have

$$(3.5)\qquad \varphi(h)=\sum_{k=0}^{N}\sum_{|J|=k}\varrho^{2k+3}\int_0^1\mu(B_h(T_1T_JT_1x))^q\,dx+e^1(h),$$

where

$$e^1(h)=\varrho^{2(N+1)}\sum_{|J|=N+1}\int_0^1\mu(B_h(T_1T_Jx))^q\,dx.$$

If $B_{h/\varrho^{2k+6}}(x)\subseteq[0,1]$, then by Proposition 2.1(i),

$$\mu(B_h(T_1T_JT_1x))=\mu((T_1T_JT_1(B_{h/\varrho^{2k+6}}(x)))=c_J\mu(B_{h/\varrho^{2k+3}}(T_1x))$$

for $|J|=k$. This implies that

$$\int_0^1\mu(B_h(T_1T_JT_1x))^q\,dx=c_J^q\int_0^1\mu(B_{h/\varrho^{2k+3}}(T_1x))^q\,dx+e_J^2(h)-\tilde e_J^2(h)$$

where

$$e_J^2(h)=\Big(\int_0^{h/\varrho^{2k+6}}+\int_{1-h/\varrho^{2k+6}}^{1}\Big)\mu(B_h(T_1T_JT_1x))^q\,dx,$$
$$\tilde e_J^2(h)=c_J^q\Big(\int_0^{h/\varrho^{2k+6}}+\int_{1-h/\varrho^{2k+6}}^{1}\Big)\mu(B_{h/\varrho^{2k+3}}(T_1x))^q\,dx.$$

Let N be the largest integer such that $0<h\leq\frac12\varrho^{2N+6}$. We can write (3.5) as

$$(3.6)\qquad \varphi(h)=\sum_{k=0}^{N}\varrho^{2k+3}\Big(\sum_{|J|=k}c_J^q\varphi\Big(\frac{h}{\varrho^{2k+3}}\Big)\Big)+e^1(h)+e^2(h),$$

where

$$(3.7)\qquad e^2(h)=\sum_{k=0}^{N}\varrho^{2k+3}\Big(\sum_{|J|=k}(e_J^2(h)-\tilde e_J^2(h))\Big).$$

It follows that

$$(3.8)\qquad \Phi^{(\alpha)}(h)=\sum_{k=0}^{N}\varrho^{-(2k+3)\alpha}\sum_{|J|=k}c_J^q\Phi^{(\alpha)}\Big(\frac{h}{\varrho^{2k+3}}\Big)+E(h),$$

where $E(h) = h^{-1-\alpha}(e^1(h) + e^2(h))$. Let $F(q,\alpha)$ and D be defined as in (1.7). Our main theorem is

THEOREM 3.2. *Let $q > 0$ and suppose (q,α) satisfies*

$$\sum_{k=0}^{\infty} \varrho^{-(2k+3)\alpha}\Big(\sum_{|J|=k} c_J^q\Big) = 1. \tag{3.9}$$

Then $(q,\alpha) \in D$, $0 < \overline{\lim}_{h\to 0^+} \Phi^{(\alpha)}(h) < \infty$ and hence $\tau(q) = \alpha$. Moreover, τ is differentiable on $(0,\infty)$.

The proof of this theorem depends on the renewal equation ([Fe], [L1], [LW]). The major technical difficulty is to prove the following lemma.

LEMMA 3.3. *Suppose $(q,\alpha) \in D$. Then there exists $\varepsilon > 0$ such that $E(h) = o(h^{\varepsilon})$ as $h \to 0^+$.*

We will postpone its proof to Section 5.

Proof of Theorem 3.2. Let N be the largest integer such that $h \le \frac{1}{2}\varrho^{2N+6}$. Then for $k \ge N+1$,

$$\Phi^{(\alpha)}\Big(\frac{h}{\varrho^{2k+3}}\Big) \le \Big(\frac{2}{\varrho^3}\Big)^{1+\alpha} \int_0^1 \mu(B_{h/\varrho^{2k+3}}(T_1 x))^q\,dx \le \Big(\frac{2}{\varrho^3}\Big)^{1+\alpha}.$$

By Proposition 2.5, D is open and $(q,\alpha) \in D$. Thus we can choose $\varepsilon > 0$ so that

$$\sum_{k=N+1}^{\infty} \varrho^{-(2k+3)\alpha}\Big(\sum_{|J|=k} c_J^q\Big) = o(h^{\varepsilon}).$$

This implies that

$$E_\infty(h) := \sum_{k=N+1}^{\infty} \varrho^{-(2k+3)\alpha} \sum_{|J|=k} c_J^q \Phi^{(\alpha)}\Big(\frac{h}{\varrho^{2k+3}}\Big) = o(h^{\varepsilon}) \quad \text{as } h \to 0^+.$$

For $0 < h < 1$, let $x = -\ln h > 0$, and $f(x) = \Phi^{(\alpha)}(e^{-x})$. Then by (3.8),

$$f(x) = \sum_{k=0}^{N} \varrho^{-(2k+3)\alpha} \sum_{|J|=k} c_J^q f(x + (2k+3)\ln\varrho) + E(e^{-x}).$$

Let ν be the measure with weight $\varrho^{-(2k+3)\alpha}\sum_{|J|=k} c_J^q$ at $-(2k+3)\ln\varrho$, $k = 0,1,2,\ldots$ Now, (3.9) implies that ν is a probability measure with support contained in $[0,\infty)$. Hence for $x > 0$,

$$f(x) = \int_0^\infty f(x-y)\,d\nu(y) + E(e^{-x}) = \int_0^x f(x-y)\,d\nu(y) + S(x), \tag{3.10}$$

where $S(x) = E(e^{-x}) + \int_x^\infty f(x-y)\,d\nu(y)$. Note that $E(e^{-x}) = o(e^{-\varepsilon x})$ as $x \to \infty$ (by Lemma 3.3), and

$$\int_x^\infty f(x-y)\,d\nu(y) = E_\infty(h)$$

is also of order $o(e^{-\varepsilon x})$ as $x \to \infty$. Since ν is a probability measure and $(q,\alpha) \in D$, the moment of ν satisfies

$$\int_0^\infty y\,d\nu(y) = -\ln\varrho \cdot \sum_{k=0}^{\infty}(2k+3)\varrho^{-(2k+3)\alpha} \sum_{|J|=k} c_J^q < \infty.$$

The renewal theorem applied to (3.10) implies that there is a nonzero bounded multiplicatively periodic function $p(h)$ of period ϱ such that $\lim_{h\to 0^+}(\Phi^{(\alpha)}(h) - p(h)) = 0$ ([Fe], [LW]). Thus, $0 < \overline{\lim}_{h\to 0^+} \Phi^{(\alpha)}(h) < \infty$ and Proposition 3.1 implies that $\tau(q) = \alpha$. The differentiability of τ follows from Proposition 2.5(iii). ■

4. Calculation of the L^q-spectrum. In this section we will study $\tau(q)$ as a solution of

$$F(q,\alpha) = \sum_{k=0}^{\infty} \varrho^{-(2k+3)\alpha}\Big(\sum_{|J|=k} c_J^q\Big) = 1.$$

By Proposition 2.3, $F(q,\alpha) = 1$ is equivalent to

$$\frac{1}{8^q}\sum_{k=0}^{\infty} \frac{\varrho^{-(2k+3)\alpha}}{4^{qk}} \sum_{|J|=k} \left(\begin{bmatrix} 1 & 1 \end{bmatrix} P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^q = 1. \tag{4.1}$$

We are going to derive a simple matrix equation for $F(q,\alpha) = 1$ when $q > 0$ is an integer. For $q \in \mathbb{N}$, we define

$$A_q^{(0)} = \begin{bmatrix} \binom{q}{0} & \binom{q}{1} & \binom{q}{2} & \cdots & \binom{q}{q} \\ 0 & \binom{q-1}{0} & \binom{q-1}{1} & \cdots & \binom{q-1}{q-1} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & \binom{1}{0} & \binom{1}{1} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

$$A_q^{(1)} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \binom{1}{0} & \binom{1}{1} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \binom{q-1}{0} & \binom{q-1}{1} & \cdots & \binom{q-1}{q-1} & 0 \\ \binom{q}{0} & \binom{q}{1} & \binom{q}{2} & \cdots & \binom{q}{q} \end{bmatrix}$$

and let $A_q = A_q^{(0)} + A_q^{(1)}$. For example,

$$A_2 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

and

$$A_3 = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}.$$

THEOREM 4.1. *For $q \in \mathbb{N}$, the equation $F(q,\alpha) = 1$ can be reduced to*

$$z^{-3} \left[\binom{q}{0} \ \ \binom{q}{1} \ \ \cdots \ \ \binom{q}{q} \right] (I - z^{-2} A_q)^{-1} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 1, \tag{4.2}$$

with $z = 2^q \varrho^\alpha$. Furthermore, $\tau(q) = \ln(z/2^q)/\ln \varrho$ where z is the largest real root of equation (4.2).

Proof. We first show that (4.2) implies that $F(q,\alpha) = 1$. Let $e_0 = [1 \ \ 0]$, $e_1 = [0 \ \ 1]$ and set

$$s_k = \sum_{|J|=k} \left([1 \ \ 1] P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^q.$$

Then by the binomial theorem,

$$s_k = \sum_{|J|=k} \left((e_0 + e_1) P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^q = \sum_{|J|=k} \left(e_0 P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e_1 P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^q$$
$$= \sum_{|J|=k} \sum_{i=0}^{q} \binom{q}{i} \left(e_0 P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^{q-i} \left(e_1 P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^i = \sum_{i=0}^{q} \binom{q}{i} \gamma_i^{(k)},$$

where

$$\gamma_i^{(k)} = \sum_{|J|=k} \left(e_0 P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^{q-i} \left(e_1 P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^i.$$

Since

$$\gamma_i^{(k+1)} = \sum_{|J|=k} \left(e_0 P_0 P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^{q-i} \left(e_1 P_0 P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^i$$
$$+ \sum_{|J|=k} \left(e_0 P_2 P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^{q-i} \left(e_1 P_2 P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^i,$$

by substituting

$$e_0 P_0 = e_0 + e_1, \quad e_1 P_0 = e_1, \quad e_0 P_2 = e_0, \quad e_1 P_2 = e_0 + e_1$$

into the above expression, we have

$$\gamma_i^{(k+1)} = \sum_{|J|=k} \sum_{j=0}^{q-i} \binom{q-i}{j} \left(e_0 P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^{q-i-j} \left(e_1 P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^{i+j}$$
$$+ \sum_{|J|=k} \sum_{j=0}^{i} \binom{i}{j} \left(e_0 P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^{q-j} \left(e_1 P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^j.$$

The matrix form of these identities for $0 \le i \le q$ is

$$\begin{bmatrix} \gamma_0^{(k+1)} \\ \gamma_1^{(k+1)} \\ \vdots \\ \gamma_q^{(k+1)} \end{bmatrix} = A_q \begin{bmatrix} \gamma_0^{(k)} \\ \gamma_1^{(k)} \\ \vdots \\ \gamma_q^{(k)} \end{bmatrix}.$$

Thus we see that

$$s_k = \left[\binom{q}{0} \ \ \binom{q}{1} \ \ \cdots \ \ \binom{q}{q} \right] \begin{bmatrix} \gamma_0^{(k)} \\ \gamma_1^{(k)} \\ \vdots \\ \gamma_q^{(k)} \end{bmatrix}$$
$$= \left[\binom{q}{0} \ \ \binom{q}{1} \ \ \cdots \ \ \binom{q}{q} \right] A_q^k \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix},$$

and

$$F(q,\alpha) = \frac{1}{8^q} \varrho^{-3\alpha} \left[\binom{q}{0} \ \ \binom{q}{1} \ \ \cdots \ \ \binom{q}{q} \right] \sum_{k=0}^{\infty} \left(\frac{1}{4^q \varrho^{2\alpha}} A_q \right)^k \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{4.3}$$

Now let $z = 2^q \varrho^\alpha$ and write the series $F(q,\alpha)$ in (4.3) as

$$F(q,\alpha) = z^{-3} \left[\binom{q}{0} \ \ \binom{q}{1} \ \ \cdots \ \ \binom{q}{q} \right] (I - z^{-2} A_q)^{-1} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

If z is the largest real solution of (4.2), then by Theorem 3.2, $\tau(q) = \alpha = \ln(z/2^q)/\ln \varrho$. If z' is also a solution of (4.2) and if $z' = 2^q \varrho^{\alpha'}$, then the uniqueness of the solution of $F(q,\alpha) = 1$ forces (q,α') to lie above D. ∎

REMARK. We can use (4.3) to find the point $(q, \tilde{\alpha}_q)$ on ∂D. Let λ be the maximum eigenvalue of A_q and let $\mathbf{v}$ be the corresponding eigenvector with $\|\mathbf{v}\| = 1$. Then each coordinate of $\mathbf{v}$ is less than 1 and the matrix A_q is nonnegative and irreducible. By the Perron–Frobenius theorem [Se], each coordinate of $\mathbf{v}$ is also strictly positive. If we let v be the smallest coordinate of $\mathbf{v}$, then

$$\lambda^k \mathbf{v} = A_q^k \mathbf{v} \le A_q^k \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \le A_q^k\left(\frac{1}{v}\mathbf{v}\right) = \lambda^k\left(\frac{1}{v}\mathbf{v}\right),$$

and hence

$$(4.4)\quad \left(1-\frac{\lambda}{4^q\varrho^{2\alpha}}\right)^{-1}\mathbf{v} \le \sum_{k=0}^{\infty}\left(\frac{1}{4^q\varrho^{2\alpha}}A_q\right)^k \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \le \left(1-\frac{\lambda}{4^q\varrho^{2\alpha}}\right)^{-1}\left(\frac{1}{v}\mathbf{v}\right).$$

If $\widetilde{\alpha}_q$ is the unique number such that $\lambda/(4^q\varrho^{2\widetilde{\alpha}_q}) = 1$, then (4.4) implies that $F(q,\widetilde{\alpha}_q) = \infty$.

We also remark that it is more convenient to write the rational function equation (4.2) into a polynomial equation $P(z) = 0$ as in the following table, which shows some numerical values obtained. See Figure 2 for graphs showing $\widetilde{\alpha}_q$ and $\tau(q)$.

q	$F(q,\alpha)=1$	$\tau(q)$	L^q-dimension
2	$z^3-2z^2-2z+2=0$	0.9923994	0.9923994
3	$z^3-2z^2-4z+2=0$	1.9794268	0.9897134
4	$z^5-2z^4-7z^3-2z+2=0$	2.9623955	0.9874652
5	$z^5-2z^4-11z^3-8z^2-20z+10=0$	3.9421547	0.9855387

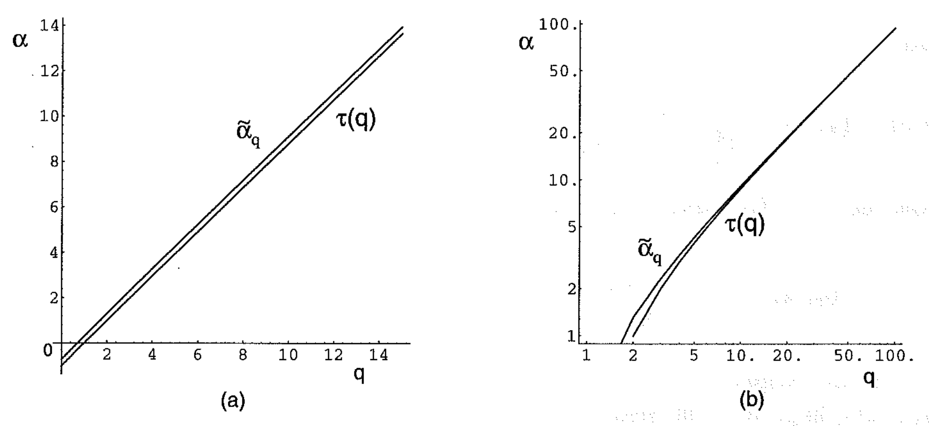

Fig. 2. (a) Graphs of $\tau(q)$ and $\widetilde{\alpha}_q$ plotted by using integer values of q for $0 < q < 15$. It can be seen that $\tau(q) < \widetilde{\alpha}_q$. (b) The same graphs are plotted using the log-log scale for larger values of q.

For any bounded regular Borel measure ν on $\mathbb{R}^d$ with compact support, we have defined $\underline{\dim}_q(\nu) = \tau(q)/(q-1)$ for $1 < q < \infty$. For $q=1$, $\tau(q) = 0$ and $\underline{\dim}_1(\nu)$ is undefined in this way. Thus we need to adjust the definition for $q = 1$ and ∞. We define

$$\underline{\dim}_\infty(\nu) = \lim_{h\to 0^+}\frac{\ln\sup\nu(Q_i(h))}{\ln h},$$

where $Q_i(h)$ is the family of h-mesh cubes on $\mathbb{R}^d$ in (1.1). We replace $\underline{\dim}_1(\nu)$ by the entropy dimension of ν. For convenience we assume that ν is a probability measure. For a finite Borel partition $\mathcal{P}$ of supp(ν), we let $|\mathcal{P}|$ be the maximum of the diameters of elements of $\mathcal{P}$. For $\delta > 0$, let

$$h(\nu,\delta) = \inf\Big\{-\sum_{A\in\mathcal{P}}\nu(A)\ln\nu(A) : \mathcal{P} \text{ a finite Borel partition of } \operatorname{supp}(\nu),\ |\mathcal{P}|\le\delta\Big\}.$$

The *entropy dimension* (or *Rényi dimension* [Ré]) of ν is defined as

$$\dim_{\rm e}(\nu) = \lim_{\delta\to 0^+}\frac{h(\nu,\delta)}{-\ln\delta}.$$

(If the limit does not exist, we replace it by $\underline{\lim}$ or $\overline{\lim}$.) We also recall that the *Hausdorff dimension* of ν is defined as

$$\dim_{\rm H}(\nu) = \inf\{\dim_{\rm H}(E) : \nu(\mathbb{R}^d\setminus E) = 0\}.$$

The following result is proved in [N].

THEOREM 4.2. *Let ν be a Borel probability measure on $\mathbb{R}^d$ with compact support. If the L^q-spectrum $\tau(q)$ of ν is differentiable at the point $q=1$, then* $\dim_{\rm H}(\nu) = \dim_{\rm e}(\nu) = \tau'(1)$.

We now return to the ICBM μ defined by $\varrho = (\sqrt{5}-1)/2$.

THEOREM 4.3.

$$\dim_{\rm H}(\mu) = \dim_{\rm e}(\mu) = \frac{1}{9\ln\varrho}\sum_{k=0}^{\infty}\sum_{|J|=k} c_J\ln c_J \quad (\approx 0.9957).$$

Proof. Observe that $F(q,\alpha)$ is differentiable on a neighborhood of $(1,0)$. By Theorem 3.2 and a direct calculation, it can be shown that

$$\tau'(1) = \frac{\sum_{k=0}^{\infty}\sum_{|J|=k} c_J\ln c_J}{\ln\varrho\sum_{k=0}^{\infty}(2k+3)\sum_{|J|=k} c_J}.$$

We can use

$$\sum_{|J|=k} c_J = \frac{1}{8}[1\quad 1]\left(\frac{1}{4}(P_0+P_2)\right)^k\begin{bmatrix}1\\1\end{bmatrix}$$

(Proposition 2.4(i)) and the derivative of $\sum_{k=0}^{\infty}\varrho^{-(2k+3)\alpha}(\sum_{|J|=k}c_J)$ at $\alpha = 0$ to conclude that $\sum_{k=0}^{\infty}(2k+3)(\sum_{|J|=k}c_J) = 9$. The result now follows from Theorem 4.2. ∎

THEOREM 4.4.

$$\underline{\dim}_{\infty}(\mu) = \left|\frac{\ln 2}{\ln \varrho}\right| - \frac{1}{2} \ (\approx 0.9404).$$

Proof. In Proposition 2.4(ii), we have shown that the maximum value of the c_J's has the same order as $1/(4\varrho)^k$ as $k \to \infty$. Hence there exists a positive constant C so that for $(q,\alpha) \in D$,

$$C\frac{\varrho^{-2k\alpha}}{(4\varrho)^{kq}} < F(q,\alpha) \quad \text{for all } k > 0.$$

For $q > 0$, $F(q,\tau(q)) = 1$ (Theorem 3.2) and hence $C(4\varrho)^{-q}\varrho^{-2\tau(q)} \le 1$. This implies that

$$\lim_{q\to\infty}\tau(q)/q \le |\ln 2/\ln\varrho| - 1/2. \tag{4.5}$$

To prove the reverse inequality, we first note that for each fixed k, the sum $\varrho^{-(2k+3)\tau(q)}(\sum_{|J|=k}c_J^q)$ tends to 0 as $q \to \infty$. In fact, if we let $a = \lim_{q\to\infty}\tau(q)/q$ and write

$$\varrho^{-(2k+3)\tau(q)}\Big(\sum_{|J|=k}c_J^q\Big) = \Big(\varrho^{-(2+3/k)k\tau(q)/q}\Big(\sum_{|J|=k}c_J^q\Big)^{1/q}\Big)^q,$$

we see that as $q \to \infty$, (4.5) and Proposition 2.4(ii) imply that

$$\lim_{q\to\infty}\varrho^{-(2+3/k)k\tau(q)/q}\Big(\sum_{|J|=k}c_J^q\Big)^{1/q} \le \varrho^{-(2+3/k)ka}\Big(\frac{1}{4(4\varrho)^k}\Big) \le \frac{1}{4\varrho^{3a}} < 1.$$

Now fix an arbitrary $k_0 \ge 1$. Then for all $q \ge 0$,

$$1 = \sum_{k=0}^{k_0-1}\varrho^{-(2k+3)\tau(q)}\Big(\sum_{|J|=k}c_J^q\Big) + \sum_{k=k_0}^{\infty}\varrho^{-(2k+3)\tau(q)}\Big(\sum_{|J|=k}c_J^q\Big),$$

and consequently there exists $q_0 = q_0(k_0)$ such that for all $q \ge q_0$,

$$\frac{1}{2} < \sum_{k=k_0}^{\infty}\varrho^{-(2k+3)\tau(q)}\Big(\sum_{|J|=k}c_J^q\Big) \le \sum_{k=k_0}^{\infty}\Big(\frac{\varrho^{-(2+3/k)\tau(q)}}{(4\varrho)^{q-1}}\Big)^k\Big(\sum_{|J|=k}c_J\Big)$$
$$\le \sup_{k\ge k_0}\Big(\frac{\varrho^{-(2+3/k)\tau(q)}}{(4\varrho)^{q-1}}\Big)^k\sum_{k=k_0}^{\infty}\Big(\sum_{|J|=k}c_J\Big).$$

(The second inequality follows from the upper bound estimate of c_J in Proposition 2.4(ii).) The last sum is less than 1 (Proposition 2.4(i)), and

hence there exists $k \ge k_0$ such that

$$\frac{1}{2} < \frac{\varrho^{-(2+3/k)\tau(q)}}{(4\varrho)^{q-1}} \quad \text{for all } q \ge q_0.$$

By letting $q \to \infty$ and then $k_0 \to \infty$, we get

$$\lim_{q\to\infty}\tau(q)/q \ge \frac{\ln(4\varrho)}{-2\ln\varrho}.$$

This is the reverse inequality of (4.5). Finally, the theorem follows by observing that $\underline{\dim}_{\infty}(\mu) = \lim_{q\to\infty}\tau(q)/q$ (see [LN1]). ∎

We remark that the same result is also obtained in [Hu] by using a completely different algebraic method.

In the multifractal formalism (see, e.g., [CM]), the spectrum $\tau(q)$ for $q < 0$ is also very interesting and has significant meaning. We have not considered the computation of $\tau(q)$ in that region yet. Also we do not know if there is an analogous calculation for P.V. numbers other than $(\sqrt{5}+1)/2$.

5. Proof of Lemma 3.3. We first establish a few basic estimations. Let $\gamma = |\ln 2/\ln\varrho|$ (≈ 1.4404).

PROPOSITION 5.1. *Let $a = \varrho$ or ϱ^2. Then there exist positive constants C_1, C_2 such that*

$$C_1 h^{\gamma}|\ln h| \le \mu(B_h(a)) \le C_2 h^{\gamma}|\ln h| \quad \textit{for } 0 < h < \varrho^2.$$

Proof. We consider the case $a = \varrho$ only. The other case $a = \varrho^2$ is the same since $\mu(B_h(\varrho)) = \mu(B_h(\varrho^2))$ (by the symmetry of μ about 1/2). Let $0 < h < \varrho^2$. Then there exists a unique integer m such that $\varrho^{m+2} \le h < \varrho^{m+1}$. The self-similar identity (2.1) implies that $\mu(B_{h/\varrho^k}(1)) = \frac{1}{2}\mu(B_{h/\varrho^{k+1}}(1))$, $1 \le k < m$, and

$$\begin{aligned}\mu(B_h(\varrho)) &= \frac{1}{2}\mu(B_{h/\varrho}(1)) + \frac{1}{2}\mu(B_{h/\varrho}(\varrho^2))\\ &= \frac{1}{2}\mu(B_{h/\varrho}(1)) + \frac{1}{2}\mu(B_{h/\varrho}(\varrho)) = \ldots\\ &= \sum_{k=1}^{m}\frac{1}{2^k}\mu(B_{h/\varrho^k}(1)) + \frac{1}{2^m}\mu(B_{h/\varrho^m}(\varrho)).\end{aligned}$$

Since $1/4 < \mu(B_{h/\varrho^m}(1)) < \mu(B_{h/\varrho^m}(\varrho)) \le 1$, we have

$$\frac{1}{4}\cdot\frac{m+1}{2^m} < \mu(B_h(\varrho)) \le \frac{m+1}{2^m}.$$

The assertion now follows by observing that $(m+1)/2^m$ is bounded above and below by constant multiples of $h^{\gamma}|\ln h|$. ∎

LEMMA 5.2. *Let $J=(j_1,\dots,j_k)$ with $j_i=0$ or 2. Then for $0<x<1$,*

$$c_J\mu(B_{h/\varrho^{2k+3}}(T_1x))\le \mu(B_h(T_1T_JT_1x))\le 2c_J\mu(B_{h/\varrho^{2k+3}}(T_1x)).$$

If further $0<x<h/\varrho^{2k+6}<1/2$, then there exist constants $C_1,C_2>0$ such that

$$\mu(B_h(T_1T_JT_1x))\le C_1 4^k c_J|\ln h|h^\gamma\le C_2 2^k|\ln h|h^\gamma.$$

Proof. The first part follows from Proposition 2.1(ii). For the second part, we note that for $0<x<h/\varrho^{2k+6}$, $B_{h/\varrho^{2k+3}}(T_1x)\subset B_{2h/\varrho^{2k+3}}(\varrho^2)$. Hence using the inequality in the first part, followed by Proposition 5.1,

$$\begin{aligned}\mu(B_h(T_1T_JT_1x))&\le 2c_J\mu(B_{2h/\varrho^{2k+3}}(\varrho^2))\\&\le C_3c_J|\ln(2h/\varrho^{2k+3})|(2h/\varrho^{2k+3})^\gamma\\&\le C_4c_J\Big(\frac{1}{\varrho^{2\gamma}}\Big)^k|\ln h|h^\gamma=C_4 4^k c_J|\ln h|h^\gamma.\end{aligned}$$

The last inequality in the lemma now follows from Proposition 2.4(ii). ∎

We now apply Lemma 5.2 to obtain the desired estimation for the error term $e^2(h)$. (See (3.7) for the definition of $e^2(h)$.)

LEMMA 5.3. *Suppose $(q,\alpha)\in D$, $q>0$. For $0<h<1$, let N be the largest integer satisfying $h<\frac12\varrho^{2N+6}$. Then there exists $\varepsilon>0$ such that $e^2(h)=o(h^{1+\alpha+\varepsilon})$ as $h\to 0^+$.*

Proof. We first estimate the part of $e^2(h)$ determined by the $e_J^2(h)$, i.e.,

$$\sum_{k=0}^N\varrho^{2k+3}\sum_{|J|=k}\int_0^{h/\varrho^{2k+6}}\mu(B_h(T_1T_JT_1x))^q\,dx. \tag{5.1}$$

(The integral over $[1-h/\varrho^{2k+6},1]$ is the same by the symmetry property of μ.)

CLAIM. *If $(q,\alpha)\in D$, then*

$$\alpha<\gamma q=|\ln 2/\ln\varrho|q,\quad \text{i.e.,}\quad 2^q\varrho^\alpha>1. \tag{5.2}$$

To see this we note that for $J=(0,\dots,0)$ with $|J|=k$, we have

$$c_J=\frac{1}{2\cdot 4^{k+1}}[1\ \ 1]\begin{bmatrix}1&k\\0&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\frac{k+1}{2\cdot 4^{k+1}}.$$

Hence for $\alpha\ge\gamma q$,

$$F(q,\alpha)\ge\sum_{k=0}^\infty\varrho^{-(2k+3)\alpha}\Big(\frac{k+1}{2\cdot 4^{k+1}}\Big)^q=\frac{\varrho^{-3\alpha}}{8^q}\sum_{k=0}^\infty\varrho^{2k(\gamma q-\alpha)}(k+1)^q=\infty,$$

proving the claim.

To estimate (5.1) we let $(q,\alpha)\in D$ with $q>0$. Use the fact that D is open to choose $\varepsilon>0$ small enough so that $(q,\alpha+\varepsilon)\in D$. It then follows from the claim above that we also have $\alpha+\varepsilon<\gamma q$. Hence (5.1) is bounded above by

$$\begin{aligned}&C_1\sum_{k=0}^N\varrho^{2k+3}\sum_{|J|=k}(4^kc_J|\ln h|h^\gamma)^q\cdot h/\varrho^{2k+6}\qquad\text{(by Lemma 5.2)}\\&\le C_2|\ln h|^qh^{1+\gamma q}\sum_{k=0}^N(4^q\varrho^{2(\alpha+\varepsilon)})^k\varrho^{-(2k+3)(\alpha+\varepsilon)}\sum_{|J|=k}c_J^q\\&\le C_2|\ln h|^qh^{1+\gamma q}\varrho^{2N(\alpha+\varepsilon-\gamma q)}\Big(\sum_{k=0}^\infty\varrho^{-(2k+3)(\alpha+\varepsilon)}\sum_{|J|=k}c_J^q\Big)\\&=o(h^{1+\alpha+\varepsilon/2}).\end{aligned}$$

(The second inequality is because $4=\varrho^{-2\gamma}$ and $\varrho^{2(\alpha+\varepsilon-\gamma q)}>1$.) It remains to estimate the part determined by $\tilde e_J^2(h)$, i.e.,

$$\sum_{k=0}^N\varrho^{2k+3}\sum_{|J|=k}c_J^q\int_0^{h/\varrho^{2k+6}}\mu(B_{h/\varrho^{2k+3}}(T_1x))^q\,dx.$$

But this sum is dominated by the one in (5.1) (by the first inequality in Proposition 2.1(ii)), and hence it has the desired convergence rate.

The lemma follows by combining the above estimations. ∎

In order to estimate $e^1(h)$, we will first establish the following proposition.

PROPOSITION 5.4. *Let $B\subseteq[-\varrho,0]$ and $J=(j_1,\dots,j_k)$ with $j_i=0$ or 2. Then*

(i) $\mu(T_1T_JT_2B)\le c_J$.

(ii) *If $J=(J',2,0,\dots,0)$, $|J'|=l$, then*

$$\mu(T_1T_JT_0B)\le\frac{k-l+2}{4^{k-l+1}}c_{J'}.$$

(iii) *If $J=(0,\dots,0)$, then*

$$\mu(T_1T_JT_0B)\le\frac{k+3}{2\cdot 4^{k+2}}.$$

A dual statement holds for $B\subseteq[1,1+\varrho]$ by interchanging the roles of T_0 and T_2.

Proof. (i) By applying (2.1) as in the proof of Proposition 2.1 we have

$$\begin{bmatrix}\mu(T_0T_2B)\\ \mu(T_1T_2B)\\ \mu(T_2T_2B)\end{bmatrix} = \begin{bmatrix}\frac{1}{4}\\ \frac{1}{4}\\ \frac{1}{4}\end{bmatrix}\mu(T_2B)$$

and hence

(5.3) $$\mu(T_1T_JT_2B) = c_J\mu(T_2B) \le c_J.$$

(ii) For any $E \subseteq [-1,0]$, (2.1) yields

(5.4) $$\begin{aligned}\mu(E+\varrho) &= \tfrac{1}{2}\mu(\varrho^{-1}E+1) + \tfrac{1}{2}\mu(\varrho^{-1}E+\varrho^2)\\ &= \tfrac{1}{4}\mu(\varrho^{-2}E+\varrho) + \tfrac{1}{4}\mu(\varrho^{-2}E+1)\end{aligned}$$

and

(5.5) $$\mu(E+1) = \tfrac{1}{2}\mu(\varrho^{-1}E+1) < \mu(E+\varrho).$$

If $E = T_0^{k-l}(B)(\subseteq [-1,0])$, then by applying (5.4) repeatedly and by using (5.5), we obtain

$$\begin{aligned}\mu(T_2E) &= \mu(\varrho^2E+\varrho) = \tfrac{1}{4}\mu(E+\varrho) + \tfrac{1}{4}\mu(E+1) = \ldots\\ &= \frac{1}{4^{k-l+1}}\mu(B+\varrho) + \sum_{i=1}^{k-l+1}\frac{1}{4^i}\mu(\varrho^{-2(i-1)}E+1)\\ &\le \frac{1}{4^{k-l+1}}\mu(B+\varrho) + \frac{k-l+1}{4^{k-l+1}}\mu(B+1)\\ &\le \frac{k-l+2}{4^{k-l+1}}\mu(B+\varrho).\end{aligned}$$

It follows from (5.3) that

$$\mu(T_1T_JT_0B) = c_{J'}\mu(T_2E) \le \frac{k-l+2}{4^{k-l+1}}c_{J'}.$$

(iii) Let $E = T_0^{k+1}(B)$. Then the above calculation (replacing $k-l$ by $k+1$) yields

$$\mu(T_1E) = \frac{1}{8}\mu(E+\varrho) + \frac{1}{8}\mu(E+1) \le \frac{k+3}{2\cdot 4^{k+2}}. \ \blacksquare$$

LEMMA 5.5. *Under the same hypotheses as in Lemma 5.3, there exists* $\varepsilon > 0$ *such that* $e^1(h) = o(h^{1+\alpha+\varepsilon})$ *as* $h \to 0^+$.

Proof. Recall from (3.5) that

$$e^1(h) = \varrho^{2(N+1)}\sum_{|J|=N+1}\int_0^1 \mu(B_h(T_1T_Jx))^q\,dx,$$

where N is the largest integer such that $0 < h < (1/2)\varrho^{2N+6}$. Consider $T_1T_JT_0$ with $|J| = N$. Since $B_{h/\varrho^{2N+5}}(x) \subseteq [-\varrho, 1+\varrho]$ for $0 \le x \le 1$, it follows from Corollary 2.2 and Proposition 5.4 (applied to $[0,1]$, $(-\varrho,0)$, $(1,1+\varrho)$ respectively) that

$$\begin{aligned}\sum_{|J|=N}\mu(B_h(T_1T_JT_0x))^q &\le \sum_{|J|=N}(2c_J)^q + \sum_{k=0}^{N-1}\sum_{|J|=k}\left(\frac{N-k+2}{4^{N-k+1}}c_J\right)^q\\ &\quad + \sum_{|J|=N}c_J^q\\ &\le C_1\sum_{k=0}^{N}\sum_{|J|=k}\left(\frac{N-k+2}{4^{N-k+1}}c_J\right)^q.\end{aligned}$$

The same estimate holds for $\sum_{|J|=N}\mu(B_h(T_1T_JT_2x))^q$.

Now we choose $\varepsilon > 0$ so that $(q, \alpha+\varepsilon) \in D$. Then $2^q\varrho^{(\alpha+\varepsilon)} > 1$ (see (5.2)) and

$$\begin{aligned}e^1(h) &\le C_1\varrho^{2(N+1)}\sum_{k=0}^{N}\sum_{|J|=k}\left(\frac{N-k+2}{4^{N-k+1}}c_J\right)^q\\ &\le C_2\frac{h}{4^{Nq}}\sum_{k=0}^{N}\varrho^{-(2k+3)(\alpha+\varepsilon)}(4^q\varrho^{2(\alpha+\varepsilon)})^k(N-k+2)^q\sum_{|J|=k}c_J^q\\ &\le C_2\frac{h}{4^{Nq}}(4^q\varrho^{2(\alpha+\varepsilon)})^N(N+2)^q\sum_{k=0}^{\infty}\varrho^{-(2k+3)(\alpha+\varepsilon)}\sum_{|J|=k}c_J^q\\ &= C_3h^{1+\alpha+\varepsilon/2}. \ \blacksquare\end{aligned}$$

Lemma 3.3 follows from the estimations in Lemmas 5.3 and 5.5.

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