On the discount sum of Bernoulli random variables

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Dedicated to Professor C.R. Rao on the occasion of his seventy-fifth birthday

Abstract

Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be i.i.d. Bernoulli random variables. For $\frac{1}{2} < \rho < 1$, let $X = \sum_{n=1}^{\infty} \rho^n \varepsilon_n$ be the discount sum and let μ_ρ be the distribution measure. It is known that if ρ^{-1} is a P.V. number, then μ_ρ is continuously singular. In this paper we use a Markov chain technique to obtain the precise L^2-dimension of such measures. In particular for $\rho = (\sqrt{5} - 1)/2$, we use a device of Strichartz et al. and the renewal equation to derive a formula for the L^p-dimension and the entropy dimension of the corresponding μ_ρ. © 1997 Elsevier Science B.V.

1. Introduction

Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be i.i.d. Bernoulli random variables, i.e., ε_n takes values 0 and 1 with probability $\frac{1}{2}$ each. For $0 < \rho < 1$, let $X = \sum_{k=1}^{\infty} \rho^k \varepsilon_k$ be the geometric discount sum of the ε_n's, and let μ_ρ be the corresponding distribution measure. μ_ρ is the convolution of the sequence $\{\frac{1}{2}(\delta_0 + \delta_{\rho^k})\}_{k=1}^{\infty}$ of point mass measures and is called an *infinitely convolved Bernoulli measure* (ICBM). Such measures possess very intricate and interesting properties concerning their absolute continuity and singularity. The first important work on this was due to Jessen and Wintner in 1935. They showed that μ_ρ is always a continuous measure, and is either purely singular or purely absolutely continuous. For $0 < \rho < \frac{1}{2}$, μ_ρ is singular and is supported by a Cantor-type set on $[0, \rho/(1 - \rho)]$, which has Hausdorff dimension $|\ln 2/\ln \rho|$. For $\rho = \frac{1}{2}$, X is a binary number with random digits 0, 1 that occur with probability $\frac{1}{2}$. It is clear that $\mu_{1/2}$ is a uniform distribution on $[0, 1]$. Furthermore, it is easy to show that if $\rho = (1/2)^{1/n}$, for $n = 1, 2, \ldots$, then μ_ρ is essentially an n-fold convolution of $\mu_{1/2}$ and is hence also absolutely continuous. In view of these examples of increasing smoothness of μ_ρ, there was a naive conjecture that for $\frac{1}{2} < \rho < 1$, μ_ρ is absolutely continuous. This was disproved by Erdös (1939) by observing that the regularity of μ_ρ depends on the algebraic properties of ρ. Recall

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that β is called an *algebraic integer* if it is a root of a polynomial L with integer coefficients and leading coefficient 1; assuming that L is of smallest degree, we call β' an *algebraic conjugate* of β if β' is another root of L. The following are two fascinating results in this direction. The first one is due to Erdös (1939) and Salem (1962), the second one is due to Garsia (1962): Suppose $\frac{1}{2} < \rho < 1$ and $\beta = \rho^{-1}$ is an algebraic integer with algebraic conjugates $\beta_1, \ldots, \beta_s$. Then

(i) β is a *Pisot–Vijayaraghavan (P.V.) number* (i.e. $|\beta| > 1$ and $|\beta_i| < 1$, $i = 1, \ldots, s$) if and only if $\hat{\mu}_\rho(\xi) \not\to 0$ as $|\xi| \to \infty$ where $\hat{\mu}_\rho$ is the Fourier transformation of μ_ρ. In particular, μ_ρ is singular by the Riemann Lebesgue lemma.

(ii) If $\beta \prod_{|\beta_i| > 1} \beta_i = 2$, then μ_ρ is absolutely continuous. Note that the assumption actually implies that $|\beta_i|$ is greater than 1 for all $i = 1, \ldots, s$ (see (Garsia, 1962 Lemma 4.7)), in contrast to case (i).

The basic idea is that for each n, the values of the sum $\sum_{k=1}^{n} \rho^k \varepsilon_k$ in case (i) repeat very often so that μ_ρ is singular, and in case (ii) the values do not repeat and are more or less equally spaced so that μ_ρ is absolutely continuous. In another direction, Erdös (1940) showed that

(iii) There exists ρ_0 sufficiently close to 1 such that for almost all $\rho_0 < \rho < 1$, μ_ρ is absolutely continuous.

He also made the conjecture that μ_ρ is absolutely continuous for almost all $\frac{1}{2} < \rho < 1$. There was no progress on this problem for a long time. Recently interest has been re-kindled, however, due to the development of the geometry of fractals, dynamical systems and iterated function systems (cf. Alexander and Yorke (1984), Bovier, Falconer (1990), Hu (to appear), Lalley, Lau and Ngai (to appear) and the references there). Let

$$S_j(x) = \rho_j x + b_j, \quad x \in \mathbb{R}^d$$

where $0 < \rho_j < 1$ and $b_j \in \mathbb{R}^d$, $j = 1, \ldots, m$, be similitudes and let $\{a_j\}_{j=1}^{m}$ be associated weights for the $\{S_j\}_{j=1}^{m}$. Then there exists a unique compact subset K and a unique probability measure μ satisfying

$$K = \bigcup_{j=1}^{m} S_j(K) \quad \text{and} \quad \mu = \sum_{j=1}^{m} a_j \mu \circ S_j^{-1}. \tag{1.1}$$

K is called a *self-similar set* and μ is called a *self-similar measure* (Falconer, 1990; Hutchinson, 1981). Both K and μ can be obtained through iteration of $\{S_j\}_{j=1}^{m}$, starting from any compact subset E in $\mathbb{R}^d$ and any probability measure on E. For the case

$$S_1(x) = \rho x, \qquad S_2(x) = \rho x + (1 - \rho), \quad x \in \mathbb{R} \tag{1.2}$$

with weights $\frac{1}{2}$ each, the self-similar measure is actually the ICBM μ_ρ, modulo a scaling factor, because

$$\hat{\mu}(\xi) = e^{i\xi/2} \prod_{n=1}^{\infty} \cos\left(\frac{1-\rho}{2} \rho^n \xi\right) \quad \text{and} \quad \hat{\mu}_\rho(\xi) = e^{i\xi/2(1-\rho)} \prod_{n=1}^{\infty} \cos\left(\frac{\rho^n}{2} \xi\right).$$

If $0 < \rho < \frac{1}{2}$, K is a Cantor-type set and is the disjoint union of the two copies $S_1(K)$ and $S_2(K)$. This nonoverlapping property of the components has been extended to the so-called *open set condition* (Hutchinson, 1981), and is one of the most fundamental concepts in the geometry of fractals. On the other hand, very few results are known without such a condition. The ICBM μ_ρ with $\frac{1}{2} < \rho < 1$ is the simplest important example of the overlapping case ($S_0[0, 1] \cap S_1[0, 1] \neq \emptyset$) and can be used as a pilot case to understand the more general situation (Lau and Ngai, to appear).

For the recent developments, Solomyak (1995) solved Erdös' conjecture following (iii). When μ_ρ is singular there are a number of ways to interpret the exponent α when $\mu(x - h, x + h) \sim c h^\alpha$ as $h \to 0$. For example, locally α can be regarded as the the so-called local dimension of μ at x, while globally, α is related to the Hausdorff dimension, entropy dimension, L^p-dimension, etc. The entropy dimension of μ_ρ when ρ^{-1} is a P.V. number was studied by Garsia (1963) and has been re-examinated (in particular when $\rho^{-1} = (\sqrt{5} + 1)/2$, the golden number) by Alexander and Yorke (1984), Alexander and Zagier (1991), Ledrappier and Porzio (preprint) and Przytycki and Unbanski (1989). The L^2-dimension of such measures has also been calculated by Lau (1991, 1993). In Lau (1991) the open set condition of the iterated function system is extended to a weaker separation condition which includes the case that ρ^{-1} is a P.V. number.

In this paper we discuss two different ways to calculate the dimensions of the singular ICBM. We first use the Markov chain approach to obtain the exact L^2-dimension of the ICBM when ρ^{-1} is a P.V. number (Theorem 3.7). This simplifies the indirect approach in Lau (1993). For $\rho = (\sqrt{5} - 1)/2$, Strichartz et al. (to appear) have an innovative idea of replacing the two maps in (1.2) by three maps that satisfy the open set condition, which yields via the self-similar identity (1.1) a complementary set of identities (see Eq. (4.2)). By using this and the renewal equation, we obtain a formula of the L^p-dimension and the entropy dimension for such μ_ρ (Theorem 4.1).

We organize the paper as follows. In Section 2 we review some definitions of the dimensions of a measure and reformulate them into the setting we need. In Section 3 we consider the L^2-dimension of the ICBM generated by the P.V. numbers and in Section 4 we describe the L^p-dimension formula of the ICBM corresponding to the golden number. The technical details concerning Section 4 are proved in Lau and Ngai (to appear). In Section 5 we discuss some open problems and remarks. We also include several graphs that approximate the density of the measures.

2. Preliminaries

Let μ be a positive bounded regular Borel measure on $\mathbb{R}$ and let $\mathrm{supp}\mu = \bigcap\{\mathbb{R} \setminus U :$ U is open and $\mu(U) = 0\}$ be the support of μ. For convenience we assume that μ has a bounded support. If μ is absolutely continuous then the density of μ is $\lim_{h \to 0} \mu(B_h(x))/2h$ for almost all x (with respect to Lebesgue measure), where $B_h(x)$ denotes the ball of radius h centered at x: in this case $\mu(B_h(x)) \sim ch$ as $h \to 0^+$. We will

define some notions of fractional density and dimension to handle the case where $\mu(B_h(x)) \sim ch^\alpha$, $0 \leqslant \alpha \leqslant 1$.

For $h > 0$, let $\{x_i = ih : i \in \mathbb{Z}\}$ be a lattice of $\mathbb{R}$ and let $E_i(h)(= E_i) = (x_{i-1}, x_i]$. For $1 \leqslant p < \infty$, let

$$S^p(h) = \sum_i \mu(E_i(h))^p$$

be the *p-variation* of μ. It is clear that when $p = 1$, $S^1(h) = \|\mu\|$ is the total variation of μ. We define the *L^p-dimension* of μ as follows: For $1 < p < \infty$,

$$\dim_p(\mu) = \lim_{h \to 0^+} \frac{\ln S^p(h)}{(p-1)\ln h},$$

and for $p = 1$,

$$\dim_1(\mu) = \lim_{h \to 0^+} \frac{\sum_i \{\mu(E_i) \ln \mu(E_i) : E_i \cap \operatorname{supp}\mu \neq \emptyset\}}{\|\mu\| \ln h}.$$

Also we let $\dim_\infty(\mu) = \lim_{h \to 0^-} \ln \sup \mu(E_i) / \ln h$. Note that $\dim_1(\mu)$ is also called the *entropy dimension* (Rényi, 1957), $\dim_2(\mu)$ the *correlation dimension*, and $\dim_p(\mu)$, $p > 1$ the *generalized Rényi dimension*. Heuristically, $\dim_1(\mu)$ is the limit of $\dim_p(\mu)$ as $p \to 1$ by L'Hôspital's rule. We can also define $\underline{\dim}_p(\mu)$, $\overline{\dim}_p(\mu)$ by replacing the lim with limsup and liminf. It is straightforward to check that for $1 < p < \infty$,

$$\underline{\dim}_p(\mu) = \sup\left\{\frac{\alpha}{p-1} : \overline{\lim_{h \to 0^+}} \frac{S^p(h)}{h^\alpha} < \infty\right\},$$

$$\overline{\dim}_p(\mu) = \inf\left\{\frac{\alpha}{p-1} : 0 < \underline{\lim_{h \to 0^+}} \frac{S^p(h)}{h^\alpha}\right\};$$

and $\dim_p(\mu) = \alpha/(p-1)$ if

$$0 < \underline{\lim_{h \to 0^+}} \frac{S^p(h)}{h^\alpha} \leqslant \overline{\lim_{h \to 0^+}} \frac{S^p(h)}{h^\alpha} < \infty. \tag{2.1}$$

The converse of (2.1) is not true in general. The L^p-dimension has a very useful alternative form in terms of the integral

$$I^p(h) = \int_{-\infty}^{\infty} \mu(B_h(x))^p \, dx.$$

Proposition 2.1. *For* $1 < p < \infty$,

$$\dim_p(\mu) = \lim_{h \to 0^+} \frac{1}{p-1}\left(\frac{\ln I^p(h)}{\ln h} - 1\right),$$

and for $p = 1$,

$$\dim_1(\mu) = \lim_{h \to 0^+} \left(\frac{\ln \int \mu(B_h(x)) \ln \mu(B_h(x)) \, dx}{\|\mu\| \ln h} - 1\right).$$

It is a direct consequence of the following lemma:

Lemma 2.2. *For* $1 \leqslant p < \infty$, $h > 0$,

$$S^p(h) \leqslant \frac{1}{h} I^p(h) \leqslant 4^p S^p(h). \tag{2.2}$$

Furthermore for $\mu(\mathbb{R}) \leqslant 1$ *and for* $\varepsilon > 0$, *there exists* h_0 *such that for* $0 < h < h_0$,

$$4\sum_i \mu(E_i(h)) \ln \mu(E_i(h)) \leqslant \frac{1}{h} \int_{-\infty}^{\infty} \mu(B_h(x)) \ln \mu(B_h(x)) \, \mathrm{d}x$$
$$\leqslant \sum_i \mu(E_i(h)) \ln \mu(E_i(h)) + \varepsilon,$$

where the sums are taken over all the $E_i(h)$ *having nonempty intersection with* supp μ.

Proof. We first observe that for $x_i = ih$,

$$\mu(E_i(h)) = \frac{1}{h} \int_{E_i(h)} \mu(E_i(h)) \, \mathrm{d}x \leqslant \frac{1}{h} \int_{E_i(h)} \mu(B_h(x)) \, \mathrm{d}x \leqslant \mu(B_{2h}(x_i)). \tag{2.3}$$

By Jensen's inequality with $\varphi(u) = |u|^p$, we have

$$S^p(h) \leqslant \sum_i \left(\int_{E_i(h)} \mu(B_h(x)) \frac{\mathrm{d}x}{h} \right)^p \leqslant \sum_i \int_{E_i(h)} \mu(B_h(x))^p \frac{\mathrm{d}x}{h} = \frac{1}{h} I^p(h).$$

On the other hand,

$$\frac{1}{h} I^p(h) = \frac{1}{h} \sum_i \int_{E_i(h)} \mu(B_h(x))^p \, \mathrm{d}x \leqslant \sum_i \mu(B_{2h}(x_i))^p \leqslant 4^p S^p(h)$$

and the first assertion follows. For the second assertion, we use Jensen's inequality with $\varphi(u) = u \ln u, u > 0$ and the last inequality of (2.3) to obtain

$$\sum_i \mu(B_{2h}(x_i)) \ln \mu(B_{2h}(x_i)) \leqslant \frac{1}{h} \int_{-\infty}^{\infty} \mu(B_h(x)) \ln \mu(B_h(x)) \, \mathrm{d}x.$$

The left inequality now follows from $\varphi(u) + \varphi(v) \leqslant \varphi(u + v)$, $u, v > 0$. For the right inequality we make use of the fact that $\varphi(u)$ is negative and decreasing for $0 < u < e^{-1}$: if all the atoms a of μ satisfy $\mu\{a\} < e^{-1}$, then there exists h_0 such that for $0 < h < h_0$, $\mu(E_i(h)) < e^{-1}$ for all i, and hence

$$\frac{1}{h} \int_{-\infty}^{\infty} \mu(B_h(x)) \ln \mu(B_h(x)) \, \mathrm{d}x = \frac{1}{h} \sum_i \int_{E_i} \mu(B_h(x)) \ln \mu(B_h(x)) \, \mathrm{d}x$$
$$\leqslant \frac{1}{h} \sum_i \int_{E_i} \mu(E_i) \ln \mu(E_i) \, \mathrm{d}x = \sum_i \mu(E_i) \ln \mu(E_i).$$

If the measure contains atoms a such that $\mu\{a\} \geqslant e^{-1}$, then it can have at most two such a. We can calculate separately the intervals that contains these a's and adjust the inequality by the ε.

3. L^2-dimension of ICBM

For convenience we let $\varepsilon_0 = 0$. Let $X = \sum_{k=0}^{\infty} \rho^n \varepsilon_n$ be the discount sum of the Bernoulli random variables and let μ_ρ be the corresponding distribution measure as in Section 1. With no confusion we will omit the subscript ρ in μ_ρ. Let $X_n = \sum_{k=0}^{n} \rho^k \varepsilon_k$ and let μ_n be the corresponding distribution. Then μ_n is the n-th convolution of $\frac{1}{2}(\delta_0 + \delta_{\rho^k})$, $1 \leq k \leq n$,

$$\mu_n\{x\} = \frac{1}{2^n} \# \left\{ \sum_{k=0}^{n} \rho^k \varepsilon_k \colon \sum_{k=0}^{n} \rho^k \varepsilon_k = x, \ \varepsilon_k = 0 \text{ or } 1, \ k = 0, \ldots, n \right\},$$

and $\{\mu_n\}_{n=1}^{\infty}$ converges to μ in distribution.

Lemma 3.1. *Let $\{\mu_n\}$ and μ be defined as above, then for $n > 0$,*

$$\mu(B_{\rho^n}(x)) \leq \mu_n(B_{\rho^n/(1-\rho)}(x)) \leq \mu(B_{2\rho^n/(1-\rho)}(x)).$$

Proof. It suffices to show that

$$\{\omega \colon X(\omega) \in B_{\rho^n}(x)\} \subseteq \{\omega \colon X_n(\omega) \in B_{\rho^n/(1-\rho)}(x)\} \subseteq \{\omega \colon X(\omega) \in B_{2\rho^n/(1-\rho)}(x)\}.$$

Indeed if $X(\omega) \in B_{\rho^n}(x)$, then

$$|X_n(\omega) - x| \leq |X_n(\omega) - X(\omega)| + |X(\omega) - x| \leq \frac{\rho^{n+1}}{1-\rho} + \rho^n = \frac{\rho^n}{1-\rho}.$$

The second identity can be obtained by the same argument.

Using this lemma we can reduce the calculation of the L^p-dimension of μ to the discrete measures μ_n.

Proposition 3.2. *There exist $C_1, C_2 > 0$ such that for any ICBM μ and for any $n > 0$,*

$$C_1 \sum_i \mu_n(E_i(\rho^n))^p \leq S^p(\rho^n) \leq C_2 \sum_i \mu_n(E_i(\rho^n))^p. \tag{2.4}$$

Consequently,

$$\dim_p(\mu) = \lim_{n \to \infty} \frac{\ln \sum_i \mu_n(E_i(\rho^n))}{n(p-1)\ln\rho},$$

and

$$\dim_1(\mu) = \lim_{n \to \infty} \frac{\sum_i \mu_n(E_i(\rho^n)) \ln \mu_n(E_i(\rho^n))}{n \ln \rho}.$$

Let $\tilde{\varepsilon}_0 = 0$ and let $\{\tilde{\varepsilon}_n\}_{n=1}^{\infty}$ be i.i.d. random variables taking values $-1, 0, 1$ with probability $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$ respectively. Let

$$Z_n = \sum_{k=0}^{n} \rho^k \tilde{\varepsilon}_k, \qquad Z = \sum_{k=0}^{\infty} \rho^k \tilde{\varepsilon}_k,$$

and let v_n and v be the corresponding measures. It is clear that $Z_n = X_n - X_n'$, $Z = X - X'$, $v_n = \mu_n * \tilde{\mu}_n$, and $v = \mu * \tilde{\mu}$ where X_n' and X' are independent, identical copies of X_n and X respectively, and $\tilde{\mu}_n(E) = \mu_n(-E)$. Let (Ω, P) be the corresponding probability space for $\{\tilde{\varepsilon}_n\}_{n=0}^{\infty}$. The following lemma is due to Garsia (1962): we include a proof here for completeness.

Lemma 3.3. *For any $c > 0$, there exists C_1, C_2 such that*

$$C_1 S^2(c\rho^n) \leqslant P(|Z_n| < c\rho^n) \leqslant C_2 S^2(c\rho^n).$$

Proof. Without loss of generality we assume that $c = 1$. Let $z_n = \sum_{k=0}^{n} \rho^k \tilde{\varepsilon}_k$, $x_n = \sum_{k=0}^{n} \rho^k \varepsilon_k$ and $x_n' = \sum_{k=0}^{n} \rho^k \varepsilon_k'$, ε_k, $\varepsilon_k' = 0$ or 1, then

$$\{z_n : |z_n| < \rho^n\} = \{x_n - x_n' : |x_n - x_n'| < \rho^n\}.$$

Let $E_i = ((i-1)\rho^n, i\rho^n]$, then

$$\sum_i \mu_n(E_i)^2 = \sum_i \int_{E_i} \left(\int \chi_{E_i}(x) \, d\mu_n(x) \right) d\mu_n(x')$$

$$= \sum_i \int_{E_i} \left(\int \chi_{E_i - x'}(x - x') \, d\mu_n(x) \right) d\mu_n(x')$$

$$\leqslant \sum_i \int_{E_i} \left(\int \chi_{\{z_n : |z_n| < \rho^n\}}(x - x') \, d\mu_n(x) \right) d\mu_n(x')$$

$$= \int_{-\infty}^{\infty} \left(\int \chi_{\{z_n : |z_n| < \rho^n\}}(x - x') \, d\mu_n(x) \right) d\mu_n(x')$$

$$= \mu_n * \tilde{\mu}_n \{z_n : |z_n| < \rho^n\} = v_n \{z_n : |z_n| < \rho^n\},$$

and the first inequality follows. On the other hand, if $|z_n| < \rho^n$, then $z_n = x_n - x_n'$. For each such $x_n \in E_k$, x_n' can only be in E_k or in the two adjacent intervals. Schwarz inequality then yields

$$v_n \{z_n : |z_n| < \rho^n\} \leqslant \sum_k \mu_n(E_k)(\mu_n(E_{k-1}) + \mu_n(E_k) + \mu_n(E_{k+1})) \leqslant 3 \sum_k \mu(E_k)^2.$$

The lemma now follows from Proposition 3.2.

Theorem 3.4. *Let μ be an ICBM defined by $\frac{1}{2} < \rho < 1$. Then for any $c > 0$,*

$$\dim_2(\mu) = \lim_{n \to \infty} \frac{\ln P(|\mathbf{Z}_n| < c\rho^n)}{n \ln \rho}.$$

Furthermore, μ is absolutely continuous with $d\mu/dx \in L^2(\mathbb{R})$ if and only if

$$\varlimsup_{n \to \infty} \frac{P(|\mathbf{Z}_n| < c\rho^n)}{\rho^n} < \infty. \tag{3.1}$$

Proof. The first assertion is a consequence of Proposition 3.2 and Lemma 3.3. The second assertion was proved by Kahane and Salem (1958) using Fourier transformation. For our alternative proof, we make use of Lemmas 3.2 and 3.3 again to reduce the statement to: μ is absolutely continuous and $d\mu/dx \in L^2(\mathbb{R})$ if and only if

$$\varlimsup_{h \to 0^+} \frac{1}{h^2} \int_{-\infty}^{\infty} |F(x + h) - F(x - h)|^2 \, dx < \infty,$$

where $F(x) = \mu(-\infty, x]$. This is a well-known theorem by Hardy and Littlewood (1928). □

In the following we will use the above theorem to calculate the L^2-dimension for the special class of ICBM generated by the P.V. numbers. Let

$$A_n = \left\{ z_n = \sum_{k=0}^{n} \rho^k \tilde{\varepsilon}_k : \tilde{\varepsilon}_k = 0, \ \pm 1, \ k = 0, \ldots, n, \ |z_n| < \frac{\rho^{n+1}}{1 - \rho} \right\}.$$

Let $\beta = \rho^{-1}$, we rescale A_n by multiplying β^n by each element of A_n and let

$$B = \bigcup_{n=1}^{\infty} \left\{ y_n = \sum_{k=0}^{n} \beta^{n-k} \tilde{\varepsilon}_k : \tilde{\varepsilon}_k = 0, \ \pm 1, \ k = 0, \ldots, n, \ |y_n| < \frac{1}{\beta - 1} \right\}.$$

Note that $\rho/(1 - \rho) = 1/(\beta - 1)$ is c in Theorem 3.4, and it has the following special property:

Proposition 3.5. *If $|y_n| \geq 1/(\beta - 1)$, then $|y_{n+1}| \geq 1/(\beta - 1)$.*

Proof. The statement is a direct consequence of the following observation:

$$|y_{n+1}| = |\beta y_n + \tilde{\varepsilon}_{n+1}| \geq \frac{\beta}{\beta - 1} - 1 = \frac{1}{\beta - 1}. \qquad \square$$

Let $Y_n = \sum_{k=0}^{n} \beta^{n-k} \tilde{\varepsilon}_k$, then $\{Y_n\}_{n=1}^{\infty}$ is a Markov chain. Proposition 3.5 says that once a path $\{y_n\}_{n=1}^{\infty}$ steps outside the barriers $\pm 1/(\beta - 1)$, then it will never return; B is hence a set of *inessential* states of the Markov chain (i.e. for $y \in B$, there exists $y' \notin B$ such that $y \to y'$ but $y' \nrightarrow y$ (Seneta, 1981, p.12). There may be finitely or infinitely many states in B.

Proposition 3.6. *For $\frac{1}{2} < \rho < 1$ and $\beta = \rho^{-1}$ a P.V. number, B is a finite set.*

Proof. An elementary proof of this proposition is given in (Lau, 1993, Theorem 2.3). It also follows from an interesting estimation of Garsia [1962, Lemma 1.51]: Let $\alpha > 1$ be an algebraic integer, $\alpha_1, \dots, \alpha_s$ be its algebraic conjugates and let σ denote the number of α_i such that $|\alpha_i| = 1$. For any nth degree polynomial L with integer coefficients a_i and height $M := \max\{|a_i|: i = 1, \dots, n\}$, if $L(\alpha) \neq 0$, then

$$|L(\alpha)| \geqslant \frac{\prod_{|\alpha_i| \neq 1} |\,|\alpha_i| - 1|}{(n+1)^\sigma (\prod_{\alpha_i > 1} |\alpha_i|)^{n+1} M^s}.$$

Now if β is a P.V. number, then for $y_n \neq y_m$ in B with $n \geqslant m$,

$$y_n - y_m = \sum_{k=0}^{n} \beta^{n-k} \tilde{\varepsilon}_k - \sum_{k=0}^{m} \beta^{m-k} \tilde{\varepsilon}'_k.$$

We use this to define a polynomial L with coefficients $\tilde{\varepsilon}_k - \tilde{\varepsilon}'_k$ (letting $\varepsilon'_k = 0$ for $m < k \leqslant n$). In this case L has height at most 2. Also from the definition of P.V. number, the set $\{\beta_i : |\beta_i| > 1\}$ is an empty set and $\sigma = 0$. It follows that

$$|L(\beta)| \geqslant \frac{\prod(1 - |\beta_i|)}{2^s} := C.$$

I.e. if $y_n \neq y_m$, then $|y_n - y_m| \geqslant C$. Since all the elements of B are bounded in between $\pm 1/(\beta - 1)$, B must be a finite set.

By using Propositions 3.5 and 3.6, we can define a substochastic matrix Q of the Markov chain $\{Y_n\}_{n=1}^{\infty}$ on the inessential states B. For each $y \in B$, we construct the corresponding row in Q as follows: let $y' = \beta y + \tilde{\varepsilon}$, $\tilde{\varepsilon} = 0, \pm 1$; if $y' \notin B$, we ignore the state; if $y' \in B$ and $\tilde{\varepsilon} = 1$ or -1, we assign probability $\frac{1}{4}$ to the y'; and if $y' \in B$ and $\tilde{\varepsilon} = 0$, we assign $\frac{1}{2}$ instead.

As an example, let $\rho = (\sqrt{5} - 1)/2$. Then $\beta = \rho^{-1} = (\sqrt{5} + 1)/2$ is a P.V. number, it satisfies $\beta^2 - \beta - 1 = 0$. By using this it is easy to show that $B = \{0, 1, \rho, -1, -\rho\}$ and the corresponding substochastic matrix is

$$Q = \frac{1}{4} \begin{bmatrix} 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 \end{bmatrix}.$$

Theorem 3.7. *For $\frac{1}{2} < \rho < 1$ such that $\beta = \rho^{-1}$ is a P.V. number, let μ be the corresponding ICBM. Let Q be the substochastic matrix defined as above and let λ be the maximal eigenvalue of Q. Then the L^2-dimension of μ is $|\ln \lambda / \ln \rho|$.*

Proof. It follows from Seneta (1981, p.121) that

$$P\left(|Z_n| < \frac{\rho^{n+1}}{1-\rho}\right) = P\left(|Y_n| < \frac{1}{\beta-1}\right) = v_0 Q^n \mathbf{1},$$

where v_0 is the row vector which equals 1 at state 0, and 0 at the other states $y \in B$. Let λ be the positive maximal eigenvalue of Q.

If Q is irreducible, then there exist a left eigenvector u_l and a right eigenvector u_r of Q corresponding to λ such that each coordinate of these two eigenvectors is strictly positive. From this we see that for some positive constants C_1, C_2,

$$C_1 v_0 Q^n u_r \leqslant v_0 Q^n \mathbf{1} \leqslant C_2 u_l Q^n \mathbf{1}.$$

This implies that $v_0 Q^n \mathbf{1}$ is of order λ^n. For $\alpha = |\ln \lambda / \ln \rho|$,

$$0 < \varliminf_{n \to \infty} \frac{P(|Z_n| < \frac{\rho^{n+1}}{1-\rho})}{\rho^{\alpha n}} \leqslant \varlimsup_{n \to \infty} \frac{P(|Z_n| < \frac{\rho^{n+1}}{1-\rho})}{\rho^{\alpha n}} < \infty, \tag{3.2}$$

and hence

$$\dim_2(\mu) = \lim_{n \to \infty} \frac{\ln P(|Z_n| < \frac{\rho^{n+1}}{1-\rho})}{n \ln \rho} = \alpha. \tag{3.3}$$

If Q is reducible, for simplicity, we assume Q can be reduced into two components

$$Q = \begin{bmatrix} Q_1 & R \\ 0 & Q_2 \end{bmatrix}$$

where Q_1 and Q_2 are irreducible, with the 0th row corresponding to state 0, which reaches all the states. It is easy to prove (3.2) from this, and (3.3) follows.

In the above example of $\rho = (\sqrt{5}-1)/2$, we can use symmetry to identify the states -1 and $-\rho$ with 1 and ρ. The corresponding matrix Q' on $\{0, 1, \rho\}$ is given by

$$Q' = \frac{1}{4} \begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}.$$

(Q' is obtained by cutting the last two rows and flipping the last two columns for states -1 and $-\rho$ to the columns for 1 and ρ.) These two matrices have the same maximal eigenvalue which is the larger positive root of the polynomial

$$(4\lambda)^3 - 2(4\lambda)^2 - 2(4\lambda) + 2 = 0. \tag{3.4}$$

By Theorem 3.7, $\dim_2(\mu) = |\ln \lambda / \ln \rho|$. The reduction to the smaller matrix also works for the other P.V. numbers (Lau, 1993). Table 1 gives a list of P.V. numbers for which $\dim_2(\mu)$ has been calculated, using Theorem 3.7. The minimal polynomial is the defining equation for $\beta = \rho^{-1}$. (The fourth row is the case of the golden number.)

Table 1

Min. Polynomial	ρ	Size of Q'	4λ	$\dim_2(\mu_\rho)$
$x^5 - x^4 \cdots - 1 = 0$	0.5086604	6	2.0573712	0.9835654
$x^4 - x^3 \cdots - 1 = 0$	0.5187901	5	2.1118009	0.9733295
$x^3 - x^2 - x - 1 = 0$	0.5436890	4	2.2226941	0.9642200
$x^2 - x - 1 = 0$	0.6180334	3	2.4811943	0.9923994
$x^3 - x^2 - 1 = 0$	0.6823275	25	2.7302333	0.9991163
$x^4 - x^3 - 1 = 0$	0.7244919	627	2.8979776	0.9999895
$x^3 - x - 1 = 0$	0.7548776	90	3.0195190	0.999990

4. The golden number

In this section we will give a formula of the L^p-dimension of the ICBM corresponding to $\rho = (\sqrt{5} - 1)/2$. The technique is to use the self-similar identity (1.1) to obtain a functional equation: the *renewal equation* (Feller, 1971). We will first use the simple case $0 < \rho < \frac{1}{2}$ to demonstrate the idea. Let S_1, S_2 be the similitudes defined as in (1.2) with $0 < \rho < \frac{1}{2}$ and let

$$\Phi^{(\alpha)}(h) = \frac{1}{h^{1+\alpha}} \int \mu(B_h(x))^p \, dx.$$

By using

$$\mu = \tfrac{1}{2}(\mu \circ S_1^{-1} + \mu \circ S_2^{-1}) \quad \text{and} \quad S_1[0,1] \cap S_2[0,1] = \emptyset, \tag{4.1}$$

we have for h sufficiently small, $B_h(S_1^{-1}(x)) \cap B_h(S_2^{-1}(x)) = \emptyset$ and

$$\Phi^{(\alpha)}(h) = \frac{1}{2^p h^{1+\alpha}} \int (\mu(B_h(\rho^{-1}x)) + \mu(B_h(\rho^{-1}(x - (1 - \rho)))))^p \, dx$$

$$= \frac{1}{2^p h^{1+\alpha}} \left(\int \mu(B_h(\rho^{-1}x))^p \, dx + \int \mu(B_h(\rho^{-1}(x - (1 - \rho))))^p \, dx \right)$$

$$= \frac{\rho}{2^{p-1} h^{1+\alpha}} \int \mu(B_{h/\rho}(x))^p \, dx \quad \text{(by a change of variables)}$$

$$= \frac{\rho^\alpha}{2^{p-1}} \Phi^{(\alpha)}\left(\frac{h}{\rho}\right).$$

If we choose $\alpha = |(p - 1)\ln 2 / \ln \rho|$, then the above multiplicative periodicity implies that

$$0 < \varliminf_{h \to 0^+} \Phi^{(\alpha)}(h) \leqslant \varlimsup_{h \to 0} \Phi^{(\alpha)}(h) < \infty,$$

and by (2.1) and Lemma 2.2, the L^p-dimension of μ is $|\ln 2/\ln \rho|$ for $p \geqslant 1$.

When $\rho = (\sqrt{5} - 1)/2$, the maps S_1, S_2 do not satisfy the disjointness condition in (4.1) and the second equality in the above calculation will not hold. To overcome this, we make use of a device of Strichartz et al. (to appear): Let

$$T_0 x = S_1 S_1 x = \rho^2 x,$$

$$T_1 x = S_2 S_1 S_1 x = S_1 S_2 S_2 x = \rho^3 x + \rho^2,$$

$$T_2 x = S_2 S_2 x = \rho^2 x + (1 - \rho^2).$$

Note that $T_i(0, 1)$, $i = 0, 1, 2$ are the three intervals $[0, \rho^2]$, $[\rho^2, \rho]$ and $[\rho, 1]$, and that $(0, 1)$ is the disjoint union of these three intervals. The disjointness condition in (4.1) is satisfied (excluding the two end points). On the other hand, the self-similar identity in (4.1) is reduced to the following "second order" self-similar identities defined by the T_i's: For $A \subseteq [0, 1]$,

$$\begin{bmatrix} \mu(T_0 T_i A) \\ \mu(T_1 T_i A) \\ \mu(T_2 T_i A) \end{bmatrix} = P_i \begin{bmatrix} \mu(T_0 A) \\ \mu(T_1 A) \\ \mu(T_2 A) \end{bmatrix}, \tag{4.2}$$

where

$$P_0 = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{8} \\ 0 & 0 & \frac{1}{4} \end{bmatrix}.$$

It follows from (4.2) that for $A \subseteq [0, 1]$ and for $J = (j_1, \dots, j_k)$ with $j_i = 0$ or 2,

$$\mu(T_1 T_J T_1 A) = c_J \mu(T_1 A) \quad \text{where } c_J = \frac{1}{4} [0, 1, 0] \, P_J \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{4.3}$$

where $T_J = T_{j_1} \cdots T_{j_k}$ and likewise for P_J. To calculate the L^p-dimension of μ, we use (4.2) to produce a functional equation for $\int_0^1 \mu(B_h(x))^p \, dx$ as in the proof of the first paragraph. Observe that

$$\int_0^1 \mu(B_h(x))^p \, dx = \left(\int_{T_0[0,1]} + \int_{T_1[0,1]} + \int_{T_2[0,1]} \right) \mu(B_h(x))^p \, dx$$

$$= \rho^2 \int_0^1 \mu(B_h(T_0 x))^p \, dx + \rho^3 \int_0^1 \mu(B_h(T_1 x))^p \, dx + \rho^2 \int_0^1 \mu(B_h(T_2 x))^p \, dx. \tag{4.4}$$

Let

$$\Phi_i^{(\alpha)}(h) = \frac{1}{h^{1+\alpha}} \int_0^1 \mu(B_h(T_i x))^p \, dx, \quad i = 0, \ 1, \ 2.$$

Then $\Phi_0^{(\alpha)}(h) = \Phi_2^{(\alpha)}(h)$ and $\Phi_1^{(\alpha)}(h)$ dominates the other two (see the second picture at the end of Section 5, the three $\Phi_i^{(\alpha)}(h)$'s are determined by μ on the three intervals $[0, \rho^2]$, $[\rho^2, \rho]$ and $[\rho, 1]$ respectively). By repeating the above argument of splitting the interval $[0, 1]$ of $\Phi_1^{(\alpha)}$ into three pieces as in (4.4), applying (4.2), (4.3) and using the change of variables, we have

$$\Phi_1^{(\alpha)}(h) = \sum_{k=0}^{\infty} \left(\sum_{|J|=k} c_J^p \right) \rho^{-(2k+3)\alpha} \Phi_1^{(\alpha)} \left(\frac{h}{\rho^{(2k-3)}} \right) + o(h^\delta). \tag{4.5}$$

To solve $\Phi_1^{(\alpha)}$, we use $x = -\ln h$ to change the equation into the following convolution equation (*renewal equation*):

$$f(x) = \int_0^x f(x-y) \, d\sigma(y) + S(x), \quad x \geqslant 0$$

where f, σ, and S are defined in the obvious way; $S(x)$ is a continuous integrable function with $\lim_{x \to \infty} S(x) = 0$. In order to have a nontrivial bounded solution, σ must be a probability measure (Feller, 1971). This implies that if we take (p, α) such that

$$F(p, \alpha) := \sum_{k=0}^{\infty} \left(\sum_{|J|=k} c_J^p \right) \rho^{-(2k+3)\alpha} = 1, \tag{4.6}$$

we have

$$0 < \varliminf_{h \to 0^-} \Phi_1^{(\alpha)}(h) \leqslant \varlimsup_{h \to 0^+} \Phi_1^{(\alpha)}(h) < \infty.$$

From this we can apply (2.1) and Lemma 2.2 to obtain

Theorem 4.1. *For $\rho = (\sqrt{5} - 1)/2$. Suppose $p > 0$, then there exists a unique α such that $F(p, \alpha) = 1$ and $\dim_p = \alpha/(p-1)$.*

The derivation of Eq. (4.5) and the proof of the theorem involves some technical estimations, the details are given by Lau and Ngai. By using the theorem we can prove the following result which is useful for computation.

Theorem 4.2. *If p is a nonnegative integer, then $F(p, \alpha) = 1$ can be reduced to a polynomial equation, and the solution α corresponds to the largest positive root of the polynomial.*

For example if $p = 2$, then the polynomial equation is $(4\rho^\alpha)^3 - 2(4\rho^\alpha)^2 - 2(4\rho^\alpha) + 2 = 0$ (the same as in (3.4)) and $\dim_2(\mu) \approx 0.99240$. If $p = 3$, then corresponding equation is $(8\rho^\alpha)^3 - 2(8\rho^\alpha)^2 - 4(8\rho^\alpha) + 2 = 0$. and $\dim_3(\mu) \approx 0.98971$.

In Section 2 we mentioned that heuristically the entropy dimension is the limit of the L^p-dimension as $p \to 1$ using the L'Hôspital's rule. In the present case the statement can be justified and we have

Proposition 4.3. *Let μ be defined as above, then*

$$\dim_1(\mu) = \frac{1}{9 \ln \rho} \sum_{k=0}^{\infty} \left(\sum_{|J|=k} c_J \ln c_J \right).$$

There are different formulae for the entropy dimension (Garsia, 1963; Alexander and Yorke, 1984; Alexander and Zagier, 1991; Ledrappier and Porzio). Using the above formula we can calculate that $\dim_1(\mu) \approx 0.99571$ which is compatible with the previous result. Also Hu (to appear) used an algebraic method and showed that $\dim_\infty(\mu) = -(1/2) - \ln 2 / \ln \rho \approx 0.9404$. It is seen that the spectrum of L^p-dimension lies in a very narrow band close to 1.

5. Remarks and open problems

The technique to reduce the overlapping case to a nonoverlapping case in the last section is quite restrictive. Besides the golden number, another P.V. number has been found to have the same property (ρ^{-1} satisfies $x^3 - 2x^2 + x - 1 = 0$), but most of them do not.

Question 1. Might one obtain an explicit formula of $\dim_p(\mu)$ as in Theorem 4.1 where μ is generated by a P.V. number?

In Section 3, the calculation of the L^2-dimension depends on Proposition 3.6 that the states in B is a finite set.

Question 2. Does this property characterize the P.V. numbers?

A positive answer would give a new characterization of the P.V. number, however a negative answer would be even more interesting because it would give a new singular ICBM.

Finally we return to the absolutely continuous case. Although Solomyak (1995) proved that for almost all $\frac{1}{2} < \rho < 1$, the corresponding ICBM is absolutely continuous, his proof does not offer any number explicitly. A very natural and important question is

Question 3. Is the ICBM corresponding to $\frac{1}{2} < \rho < 1$ where ρ is a rational number absolutely continuous?

In the following we draw the graphs (Figs. 1 and 2) of the approximate densities, $\mu(x-h, x+h)/2h$ for h small, of the ICBMs corresponding to $\rho = \frac{1}{3}$, $(\sqrt{5}-1)/2$, $\frac{2}{3}$ and $\frac{3}{4}$. It is seen that in the case $\rho = \frac{1}{3}$, the measure is very singular and is concentrated on the Cantor set in a uniform manner; this can be used to explain that the L^p-dimension
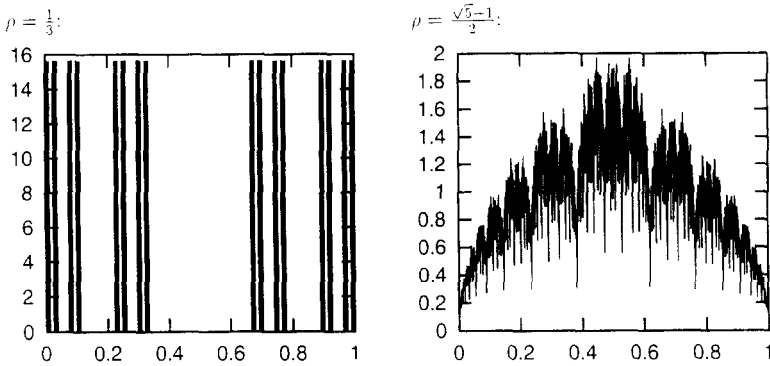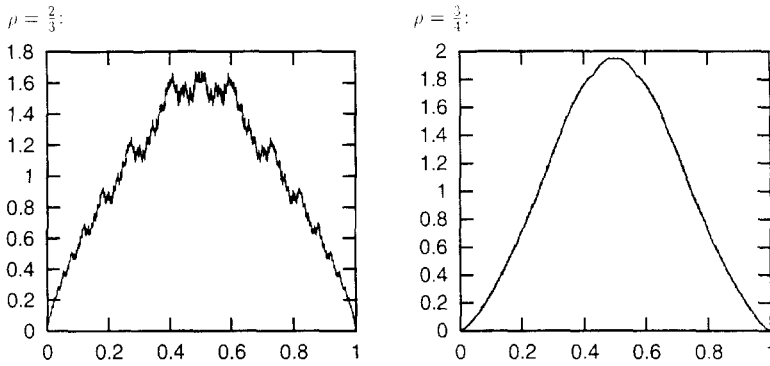
$\rho = \frac{1}{3}$:

$\rho = \frac{\sqrt{5}-1}{2}$:

Fig. 1.

$\rho = \frac{2}{3}$:

$\rho = \frac{3}{4}$:

Fig. 2.

$(= |\ln 2 / \ln \rho|)$ is independent of p (see the first paragraph of Section 4). For the case $\rho = (\sqrt{5} - 1)/2$, μ is also singular and is concentrated on a dense subset of $[0, 1]$, which is composed of sets of different local dimension; it is a *multifractal measure* (Lau and Ngai, to appear). For the cases $\rho = \frac{2}{3}$ and $\rho = \frac{3}{4}$, the pictures suggested that both measures are absolutely continuous, as is asked in Question 3.

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