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A Weighted Tauberian Theorem

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ABSTRACT. *We prove a Tauberian theorem of the form* $\phi * g.x) \sim p.x)w.x)$ *as* $x \to \infty$, *where* $p.x)$ *is a bounded periodic function and* $w.x)$ *is a weighted function of power growth. It can be used to study the weighted average of the form* $.T^{\alpha}.\ln T)^{\beta})^{-1} \int_0^T h.t)\, dt$.

1. Introduction

Tauberian theorems concern the asymptotic behavior of functions (or sequences) deduced from the behavior of their averages. The most celebrated Tauberian theorem is due to Wiener [W2] and is as follows.

Theorem 1.1.

For $\phi \in L^{\infty}.\mathbb{R})$, *the relation* $\lim_{x\to\infty} \phi * g.x) = 0$ *holds true for all* $g \in L^1.\mathbb{R})$ *whenever it holds true for some* $f \in L^1.\mathbb{R})$ *such that the Fourier transformation* $\hat{f}.\xi) \neq 0$ *for all* $\xi \in \mathbb{R}$.

This theorem is an important consequence of the more general treatment of the translation invariant subspaces of $L^1.\mathbb{R})$ (see, e.g., [B], [R], or [T]). It can be reformulated on the multiplicative group $\mathbb{R}^+$ by using the expression $\lim_{T\to\infty} \int_0^{\infty} \phi.Tx)g.x)\, dx$. In particular, if $g.x) = \chi_{[0,1]}$, then the limit becomes

$$\lim_{T\to\infty} \int_0^1 \phi.Tx)\, dx = \lim_{T\to\infty} \frac{1}{T} \int_0^T \phi.x)\, dx,$$

which is the most elementary average. This average was actually Wiener's original motivation to develop his Tauberian theorem [W1, W2], by which he proved the Wiener–Plancherel theorem on the class of functions F with bounded quadratic averages ($\limsup_{T\to\infty} \frac{1}{T} \int_0^T |F.x)|^2\, dx < \infty$) and their Fourier transformations [W1].

There are interesting cases where a function $\phi.x)$ (or its average) behaves like a periodic function at large x. For example, the solution ϕ of the *renewal equation*

$$\phi.x) = \int_0^x \phi.x - y)\, d\mu.y) + S.x), \qquad x > 0$$

is asymptotically a periodic function and the period depends on the support of the probability measure μ [F, Chapter 11]. Another important class of examples appears in the recent study of "self-similarity". It is known that the Fourier transformation $\hat{\mu}$ of the Cantor measure behaves chaotically as $|\xi| \to \infty$. On the other hand, Strichartz [S1] proved that the weighted quadratic average

$$\varphi.T) = \frac{1}{T^{1-\alpha}} \int_{-T}^{T} |\hat{\mu}.\xi)|^2\, d\xi, \qquad .1.1)$$

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where $\alpha = \ln 2/\ln 3$ is the dimension of μ, is asymptotically a multiplicative periodic function. This phenomenon holds for more general self-similar measures, and the proof is via an extension of the above Tauberian theorem [L, LW, S2]. Further investigation of such averages can be found in [JRS], where numerical solutions and open problems are presented. The self-similarity and the Tauberian theorem also play a role in the study of compactly supported L^2-solutions ϕ of the *two-scale dilation equation* [D]

$$\phi.x) = \sum_{n=1}^{N} c_n \phi.2x - n).$$

In [LWM] it is proved that (using Corollary 4.5 here)

$$\varphi.T) = \frac{1}{T^{1-\alpha}.\ln T)^{\beta}} \int_{-T}^{T} |\hat{\phi}.\xi)|^2 \, d\xi \qquad .1.2)$$

is asymptotically multiplicatively periodic, where the α is the Sobolev exponent of ϕ and the β is related to the multiplicity of the eigenvalue of a matrix associated with the coefficients $\{c_n\}$ of this equation.

In this note our main purpose is to provide a general Tauberian theorem that covers all the above cases, namely, a Tauberian theorem of the form

$$\lim_{x\to\infty} \left(\frac{\phi * g.x)}{w.x)} - p.x) \right) = 0, \qquad .1.3)$$

where $p.x)$ is a bounded periodic function and $w.x)$ is certain weighted function; this will include the cases of x^{α} or $x^{\alpha}.\log x)^{\beta}$, $\alpha, \beta \geq 0$. To prove such a theorem (Theorem 3.3), we adapt the traditional approach [R] by first obtaining a Tauberian theorem on the translation invariant subspaces of the weighted space $L^1.w)$ (Theorem 3.2). Once (1.3) is established we can easily derive corollaries that include convolutions with measures and on the multiplicative group $\mathbb{R}^+$.

An example at the end of §3 shows that some restrictions on the growth of w are necessary.

We remark that in [BBE], Wiener's Tauberian Theorem was extended to $\mathbb{R}^d$ and used to prove the Wiener–Plancherel theorem on $\mathbb{R}^d$. It is likely that the present weighted consideration can be carried to such a setting. We also remark that there is another kind of weighted Tauberian theorem that was investigated in [Bi] and [F] (Beurling's Tauberian theorem) and has important applications to the central limit theorem.

2. The Weighted Functions

Let Ω be the class of continuous functions $w : \mathbb{R} \to \mathbb{R}^+$ such that for any $x, y \in \mathbb{R}$,

i. $w.0) \geq 1$, $w.x) = w.-x)$;

ii. $w.x + y) \leq w.x)w.y)$, $w.xy) \leq w.x)w.y)$;

iii. $\lim_{x\to\infty} \frac{w.x)}{w.x+1)} = 1$ and there exist $K > 0$ and an integer $n > 0$ such that $x^{-n}w.x)$ is decreasing for $x > K$.

Some typical examples of this class of functions are

$$w.x) = a + |x|^{\alpha} \quad \text{and} \quad a + .\log^{+} |x|)^{\beta},$$

where $\alpha, \beta \geq 0$ and $a > 1$ is sufficiently large. It is easy to check that $w.x) \geq 1$ for $x \in \mathbb{R}$, and if $w_1, w_2 \in \Omega$, then $w_1 w_2 \in \Omega$.

Proposition 2.1.

Suppose u is a continuous function on $\mathbb{R}$ that satisfies i and iii and there exists an $M > 0$ such that ii holds for all $|x|, |y| > M$. Then there exists $w \in \Omega$ such that $\lim_{x\to\infty} u.x)/w.x) = 1$.

Proof. Let $a > 1$ be large enough so that $u.x + y), u.xy) \le a^2 - a$ for all $|x|, |y| \le M$. Let $w.x) = a + u.x)$. Then it is straightforward to show that $w \in \Omega$ and both u and w have the same property. □

We use $L^1.w)$ to denote the class of f such that $\|f\|_1 := \int |f.x)|w.x)\,dx < \infty$ and $L^\infty.w^{-1})$ the class of real-valued f such that $\|f\|_\infty := \text{ess sup}_x |f.x)/w.x)| < \infty$.

Proposition 2.2.

Let $w \in \Omega$. Then $L^1.w)$ is a Banach algebra and its dual is $L^\infty.w^{-1})$. Moreover if $f \in L^1.w)$ and $\phi \in L^\infty.w^{-1})$, then $\|\phi * f\|_\infty \le \|\phi\|_\infty \|f\|_1$.

Proof. The first statement depends upon the fact that $w.x + y) \le w.x)w.y)$. The last inequality follows from

$$|\phi * f.x)| \le \int |\phi.x - y)f.y)| \frac{w.x)w.y)}{w.x - y)}\,dy \le w.x)\|\phi\|_\infty \|f\|_1. \qquad \square$$

The following is the key lemma of our Tauberian theorems. It is a modification of [R, Lemma 9.2].

Lemma 2.3.

Let $w \in \Omega$ and $f \in L^1.w)$. Then for any $\epsilon > 0$ and any fixed ξ_0, there exists an $h \in L^1.w)$ such that $\|h\|_1 < \epsilon$ and

$$\hat{h}.\xi) = \hat{f}.\xi_0) - \sum_{k=1}^{n+1} .-1)^{k+1} \binom{n+1}{k} \hat{f}.k.\xi - \xi_0) + \xi_0) \qquad .2.1)$$

for all ξ in some neighborhood of $\xi_0 \in \mathbb{R}$. Here n is the integer associated with w in iii.

Proof. Without loss of generality we assume that $\xi_0 = 0$, $\hat{f}.0) = 1$. We can choose a rapid decreasing C^∞-function g such that $\hat{g}.\xi) = 1$ for all ξ in some neighborhood of 0. For $\lambda > 0$, let

$$g_\lambda.x) = \frac{1}{\lambda} g\left(\frac{x}{\lambda}\right) \quad \text{and} \quad h_\lambda.x) = g_\lambda.x) - g_\lambda * \sum_{k=1}^{n+1} .-1)^{k+1} \binom{n+1}{k} f_k.x).$$

Then

$$\widehat{h_\lambda}.\xi) = \hat{g}.\lambda\xi)\left(1 - \sum_{k=1}^{n+1} .-1)^{k+1} \binom{n+1}{k} \hat{f}.k\xi)\right),$$

which satisfies (2.1) in some neighborhood of 0.

We claim that $\|h_\lambda\|_1 \to 0$ as $\lambda \to \infty$. Once this is established, the lemma then follows by taking $h = h_\lambda$ for λ large enough. To prove the claim, we observe that

$$\begin{aligned} \|h_\lambda\|_1 &= \int \left| g_\lambda.x) - \int \sum_{k=1}^{n+1} .-1)^{k+1} \binom{n+1}{k} \frac{1}{k} g_\lambda.x - y) f\left(\frac{y}{k}\right) dy \right| w.x)\,dx \\ &= \int \left| \int f.y)\left(g_\lambda.x) - \sum_{k=1}^{n+1} .-1)^{k+1} \binom{n+1}{k} g_\lambda.x - ky) \right) dy \right| w.x)\,dx \\ &\le \int |f.y)| \left(\int \left| \sum_{k=0}^{n+1} .-1)^k \binom{n+1}{k} g\left(x - \frac{ky}{\lambda}\right) \right| w.\lambda x)\,dx \right) dy \\ &= I_1 + I_2, \end{aligned}$$

where I_1 is the integral over $\{y : |y| < \delta\lambda\}$ and I_2 is the integral over $\{y : |y| \geq \delta\lambda\}$. If $0 < \delta < 1$ and $|y| < \delta\lambda$, then the mean value theorem implies

$$\left|\sum_{k=0}^{n+1} .-1)^k \binom{n+1}{k} g\left(x - \frac{ky}{\lambda}\right)\right| \leq \left|\frac{y}{\lambda}\right|^{n+1} \tau.x),$$

where $\tau.x) = \max\{|g^{.n+1/}.u)| : x - 1 \leq u \leq x + 1\}$. Note that τ is still rapidly decreasing and belongs to $L^1.w)$. The decreasing property of $x^{-n}w.x)$ in iii implies that for $|y| > K$, $\lambda^{-n}w.\lambda) \leq y^{-n}w.y)$; hence,

$$\begin{aligned}
I_1 &= \int_{|y|<\delta\lambda} |f.y)|\left(\int \tau.x)w.x)\,dx\right)\left|\frac{y}{\lambda}\right|^{n+1} w.\lambda)\,dy \\
&= \|\tau\|_1\left(\int_{\substack{|y|<\delta\lambda,\\ |y|,|\lambda|<K}} + \int_{\substack{|y|<\delta\lambda,\\ |y|<K,\lambda\geq K}} + \int_{K\leq|y|<\delta\lambda}\right)|f.y)|\left|\frac{y}{\lambda}\right|^{n+1} w.\lambda)\,dy \\
&\leq \delta\|\tau\|_1\left(\delta^n w.K)\int_{|y|<K}|f.y)|\,dy + \max_{\lambda>K}\left\{\frac{w.\lambda)}{\lambda^n}\right\}\int_{|y|<K}|f.y)||y^n\,dy\right. \\
&\quad \left. + \int_{|y|<\delta\lambda}|f.y)|w.y)\,dy\right) \\
&\leq C\,\delta.
\end{aligned}$$

To establish I_2 we note that for $|y| \geq \delta\lambda$ and any x

$$\sum_{k=0}^{n+1}\binom{n+1}{k} w.\lambda|x| + k|y|) \leq \sum_{k=0}^{n+1}\binom{n+1}{k}\left(w\left(\frac{1}{\delta}\right)w.y)w.x) + w.k)w.y)\right) \leq Cw.x)w.y),$$

where $C = \sum_{k=0}^{n+1}\binom{n+1}{k}\big(w(\frac{1}{\delta}) + w.k)\big)$. Thus

$$\begin{aligned}
I_2 &\leq \int_{|y|\geq\delta\lambda}|f.y)|\left(\int |g.x)|\sum_{k=0}^{n+1}\binom{n+1}{k} w.\lambda|x| + k|y|)\,dx\right)dy \\
&\leq C\,\|g\|_1\int_{|y|\geq\delta\lambda} f.y)w.y)\,dy,
\end{aligned}$$

which converges to 0 as $\lambda \to \infty$. The claim now follows from the two estimates on the integrals I_1 and I_2. $\square$

3. The Tauberian Theorems

We first formulated the Tauberian theorem in terms of the spectra of ϕ and f.

Theorem 3.1.

Let $w \in \Omega$. Let $\phi \in L^\infty.w^{-1})$ and Y be a subspace of $L^1.w)$. If $\phi * f = 0$ for all $f \in Y$, then

$$\operatorname{supp}\hat{\phi} \subseteq \bigcap\{\xi : \hat{f}.\xi) = 0 \textit{ for all } f \in Y\},$$

where supp $\hat{\phi}$ *is the support of the tempered distribution* $\hat{\phi}$.

The proof is the same as in [R, Theorem 9.3], using Lemma 2.3 (replacing [R, Lemma 9.2]) to localize $\hat{f}$ on a given neighborhood and that f has only small perturbation. By using the same argument as in [R, Theorem 9.4], we have the following Tauberian theorem expressed in translation invariant subspaces.

Theorem 3.2.

Let $w \in \Omega$. Suppose Y is a closed translation invariant subspace in $L^1.w)$ generated by the translates of f. Then $Y = L^1.w)$ if and only if $\hat{f}.\xi) \neq 0$ for all ξ in $\mathbb{R}$.

Let P_a denote the class of bounded periodic functions with period a.

Theorem 3.3.

Let $w \in \Omega$. Let $\phi \in L^\infty.w^{-1})$, and assume f in $L^1.w)$ is such that $\hat{f}.\xi) \neq 0$ for all $\xi \in \mathbb{R}$. Suppose

$$\lim_{x\to\infty}\left(\frac{1}{w.x)}\phi * f.x) - p.x)\right) = 0$$

for some $p \in P_a$. Then for any $g \in L^1.w)$, there exists $q \in P_a$ such that

$$\lim_{x\to\infty}\left(\frac{1}{w.x)}\phi * g.x) - q.x)\right) = 0. \qquad .3.1)$$

Moreover, the Fourier coefficients $\{a_k\}$ and $\{b_k\}$ of p, q, respectively, are related by

$$b_k = a_k \frac{\hat{g}.2\pi k/a)}{\hat{f}.2\pi k/a)}, \qquad k \in \mathbb{N}. \qquad .3.2)$$

Proof. For convenience we assume that $a = 2\pi$. Let

$$Y = \left\{g \in L^1.w) : \lim_{x\to\infty}\left(\frac{1}{w.x)}\phi * g.x) - q.x)\right) = 0 \quad \text{for some} \quad q \in P_a\right\}.$$

Clearly Y is translation invariant. To show that Y is closed, let $\{g_n\} \subset Y$ with $g_n \to g \in L^1.w)$, and let $\{q_n\}$ be the corresponding periodic functions in P_a. Then for any $\epsilon > 0$ and any m, n there exists k_0 such that for $k > k_0$ and for any x in $[0, 2\pi]$

$$\begin{aligned} \left|q_m.x) - q_n.x)\right| &= \left|q_m.x + 2\pi k) - q_n.x + 2\pi k)\right| \\ &\leq \frac{1}{w.x + 2\pi k)}|\phi * g_m.x + 2\pi k) - \phi * g_n.x + 2\pi k)| + \epsilon \\ &\leq \left\|\phi\right\|_\infty \left\|g_m - g_n\right\|_1 + \epsilon. \end{aligned}$$

This implies that $\{q_n\}$ is a Cauchy sequence in $L^\infty[0, 2\pi]$ and converges to a bounded period function q. It is straightforward to show that $\lim_{x\to\infty} .\frac{1}{w.x/}\phi * g.x) - q.x)) = 0$ so that $g \in Y$. Then Theorem 3.2 implies that $Y = L^1.w)$ and the first part of the theorem holds.

To determine the Fourier coefficients of the periodic function q, we first observe that

$$b_k = \frac{1}{2\pi}\int_0^{2\pi} q.x)e^{-ikx}\,dx = \lim_{l\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\frac{\phi * g.x + 2\pi l)}{w.x + 2\pi l)}e^{-ikx}\,dx.$$

Since $L^1.w)$ equals the closed subspace spanned by the translates of f, we can find a sequence $\{h_n\} \subset L^1.w)$ such that $\{h_n * f\}$ converges to g in $L^1.w)$. Hence for $x \in [0, 2\pi]$ and n, l positive integers,

$$\left|\int_0^{2\pi}\frac{\phi * .h_n * f - g).x + 2\pi l)}{w.x + 2\pi l)}e^{-ikx}\,dx\right| \leq 2\pi \left\|\phi\right\|_\infty \left\|h_n * f - g\right\|_1.$$

We apply this to interchange the limit in the following calculation and thus complete the proof as

$$b_k = \lim_{l\to\infty} \lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} \frac{.\phi * h_n * f).x + 2\pi l)}{w.x + 2\pi l)} e^{-ikx}\, dx$$

$$= \lim_{n\to\infty} \lim_{l\to\infty} \frac{1}{2\pi} \int_0^{2\pi} \frac{.\phi * h_n * f).x + 2\pi l)}{w.x + 2\pi l)} e^{-ikx}\, dx$$

$$= \lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} h_n * p.x) e^{-ikx}\, dx$$

$$= \lim_{n\to\infty} \hat{h}_n.k) a_k = \frac{\hat{g}.k)}{\hat{f}.k)} a_k. \qquad \square$$

Remark. The referee suggested the following proof. Assume $a = 2\pi$ and let $h_n.t) = \phi.2\pi n + t)/w.2\pi x)$. Since

$$\left| \frac{h_n.t)}{w.t)} \right| = \left| \frac{\phi.2\pi n + t)}{w.2\pi n) w.t)} \right| \le \left| \frac{\phi.2\pi n + t)}{w.2\pi n + t)} \right| < \infty \quad \text{uniformly on } n \text{ and } t,$$

$\{h_n\}_n$ is a bounded family in $L^\infty.w^{-1})$ and has a weak* limit ψ in $L^\infty.w^{-1})$ as $n \to \infty$. It follows from the limit assumption that

$$\int_{-\infty}^{\infty} \psi.s - t) f.t)\, dt = p.s).$$

Since p has spectrum in $\mathbb{Z}$, we can apply the same argument as in Theorem 3.1 (i.e., [R, Theorem 9.3]), again using Lemma 2.3 and $\hat{f}.\xi) \neq 0$ to show that the spectrum of ψ is contained in $\mathbb{Z}$. Thus ψ is a bounded periodic function. Let $F.s) = \sum_{k=-\infty}^{\infty} f.s + 2\pi k)$ be the periodization of f. Then

$$\int_0^{2\pi} \psi.s - t) F.t)\, dt = p.s)$$

and $\hat{p}.k) = \hat{F}.k)\hat{\psi}.k)$, $k \in \mathbb{Z}$. This implies that ψ is the unique limit point of $\{h_n\}_n$ as $n \to \infty$. Now for any $g \in L^1.w)$ we have

$$\int_{-\infty}^{\infty} \psi.s - t) g.t)\, dt = q.s),$$

which implies (3.1). That $\hat{F}.k)$ (Fourier coefficient) $= \hat{f}.k)$ (Fourier transformation) yields the relationship of the Fourier coefficients in (3.2). $\square$

Corollary 3.4.
Let $w \in \Omega$. Suppose $\phi \in L^\infty.w^{-1})$ and $f \in L^1.w)$ is such that $\hat{f}.\xi) \neq 0$ for all $\xi \in \mathbb{R}$. If

$$\lim_{x\to\infty} \frac{1}{w.x)} \phi * f.x) = \hat{f}.0)c$$

for some $c \in \mathbb{R}$, then

$$\lim_{x\to\infty} \frac{1}{w.x)} \phi * g.x) = \hat{g}.0)c \quad \textit{for all } g \in L^1.w).$$

In the following we show that some growth restrictions on the weighted function w in Theorems 3.1, 2, 3 are necessary.

Let $w.x) = e^x$. Then $L^1.w)$ is a Banach algebra, but $w \notin \Omega$. Consider

$$\phi.x) = e^x; \qquad f.x) = \begin{cases} e^{-2x} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$

Then

$$\int_{-\infty}^{\infty} f.x)e^{-i\xi x}\,dx = \frac{1}{.2 + i\xi)} \neq 0 \quad \text{for all } \xi \in \mathbb{R}$$

and

$$\phi * f.x) = \int_0^{\infty} e^{x-y}e^{-2y}\,dy = \frac{1}{3}e^x. \qquad .3.3)$$

Let h be a bounded function on $\mathbb{R}$ with the property that h vanishes for $x < 0$ and

$$\int h.y)e^{-2y}\,dy \neq 0, \qquad \int h.y)e^{-3y}\,dy = 0.$$

If $g.x) = h.x)e^{-2x}$, then $g \in L^1.w)$, $\hat{g}.0) = \int g.y)\,dy \neq 0$, and

$$\phi * g.x) = \int g.y)e^{x-y}\,dy = e^x \int h.y)e^{-3y}\,dy = 0. \qquad .3.4)$$

Since (3.3) and (3.4) are inconsistent with Corollary 3.4, it follows that the theorems in this section do not hold for $L^1.w)$ where w has the exponential growth.

4. Some Extensions

We can extend Theorem 3.3 to include convolutions of measures as in Wiener's second Tauberian Theorem [W2; T, Theorem 7.6]. Let W be the class of continuous functions f on $\mathbb{R}$ such that

$$\|f\| := \sum_{k=-\infty}^{\infty} \left(\sup_x |.f\chi_{[k,k+1/)}.x)| \right) w_A.k) < \infty,$$

where $w_A.k) = \int_k^{k+1} w.\xi)\,dt$. It is easy to show that W^*, the dual of W, is the class of regular Borel measures μ satisfying

$$\|\mu\| := \sup_k |\mu|[k, k+1)/w_A.k) < \infty.$$

Note that by assumption iii for $w \in \Omega$, we can actually use $w.k)$ instead of $w_A.k)$.

Theorem 4.1.

If we replace $L^1.w)$ and $L^\infty.w^{-1})$ in Theorem 3.3 by W and W^*, then the same conclusion holds.

The proof is essentially the same as Theorem 3.3, starting from a straight forward modification of Lemma 2.3 (see [T, Theorem 7.6]).

Next we give a very useful criterion (see Corollaries 4.4 and 4.5) when the measure μ is not known to be in W^* a priori.

Corollary 4.2.

Let $w \in \Omega$ and let μ be a positive regular Borel measure on $\mathbb{R}$ such that $\overline{\lim}_{k\to-\infty} \mu[k, k+1) < \infty$. Suppose there exists $f \in W$, $f \geq 0$, such that $\hat{f}.\xi) \neq 0$ for all $\xi \in \mathbb{R}$ and

$$\lim_{x\to\infty} \left(\frac{\mu * f.x)}{w.x)} - p.x) \right) = 0 \qquad .4.1)$$

for some $p \in P_a$. Then for any $g \in W$, there exists $q \in P_a$ such that

$$\lim_{x\to\infty}\left(\frac{\mu * g.x)}{w.x)} - q.x)\right) = 0$$

and the Fourier coefficients of p and q are related as in Theorem 4.1.

Proof. We need only show that $\overline{\lim}_{k\to\infty}\mu[k, k+1)/w.k) < \infty$; this combined with the given condition $\overline{\lim}_{k\to-\infty}\mu[k, k+1) < \infty$ implies that $\mu \in W^*$ and then we can apply Theorem 3.5. To prove this we let $h.x) = e^{-x^2}$ and $f_1 = f * h$. Then $f_1 \in W$ and $\hat{f}_1.\xi) = \hat{f}.\xi)\hat{h}.\xi) \neq 0$ for all $\xi \in \mathbb{R}$. By the dominated convergence theorem we obtain

$$\lim_{x\to\infty}\left(\frac{\mu * f_1.x)}{w.x)} - p * h.x)\right) = 0.$$

It follows that $\mu * f_1.x)/w.x),\ x > 0$, is bounded, say by C. Hence for $k > 0$,

$$\begin{aligned} C &\geq \frac{1}{w.k)}\int_{-\infty}^{\infty} f_1.k-y)\,d\mu.y) \geq \frac{1}{w.k)}\int_{k}^{k+1} f_1.k-y)\,d\mu.y) \\ &\geq \inf\{f_1.y) : -1 \leq y \leq 0\}\frac{\mu[k, k+1)}{w.k)} \\ &\geq C'\,\frac{\mu[k, k+1)}{w.k)}, \end{aligned}$$

where $C' = \inf\{f_1.y) : -1 \leq y \leq 0\} > 0$. This implies that $\overline{\lim}_{k\to\infty}\mu[k, k+1)/w.k) < \infty$ and the proof is complete. □

For many applications it is useful to include discontinuous functions f and g. A way to handle this case is to use the space $\widetilde{W}$ of locally Riemann integrable functions f such that

$$\sum_{k=-\infty}^{\infty} \text{ess sup}_x |.f\chi_{[k,k+1)}).x)|w.k) < \infty \qquad .4.2)$$

(see [W2, T, Chapter 7]).

Corollary 4.3.

The f and g in Corollary 4.2 can be replaced by f and $g \in \widetilde{W}$.

Proof. Let $f \in \widetilde{W}$ be as in Corollary 4.2. By convolving with e^{-x^2}, we can actually assume that f is continuous, hence in W (see the proof in the last corollary) and so by Theorem 4.1,

$$\lim_{x\to\infty}\left(\frac{\mu * g.x)}{w.x)} - q.x)\right) = 0$$

for all $g \in W$. To extend this to all the $g \in \widetilde{W}$, we make use of an equivalent definition of (4.2) [T, Chapter 7]. If $g \in \widetilde{W}$, then there exist $\{g_i\}, \{h_i\} \subset W$ such that the sequences $g_i \searrow g$, $h_i \nearrow g$ and $\lim_{i\to\infty}\int |g_i.x) - h_i.x)|w.x)\,dx = 0$. If $\{q_j\}$ and $\{r_j\}$ denote the corresponding periodic functions, then $q_j \searrow q, r_j \nearrow r$ for some periodic functions q, r of period a. We observe that $q \geq r$. An application of Corollary 4.2 yields that

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} .q.x) - r.x))e^{ikx}\,dx &= \frac{1}{2\pi}\lim_{j\to\infty}\int_0^{2\pi} .q_j.x) - r_j.x))e^{ikx}\,dx \\ &= \lim_{j\to\infty} a_k\frac{\hat{g}_j.2\pi k/a) - \hat{h}_j.2\pi k/a)}{\hat{f}.2\pi k/a)} \\ &= 0. \end{aligned}$$

This implies that $q = r$. For such a q it is easy to show directly that $\lim_{x\to\infty}.\frac{\mu*g.x/}{w.x/} - q.x)) = 0$. □

Finally we like to express the Tauberian theorem on the multiplicative group $\mathbb{R}^+$. For simplicity we just write down a special case. Let

$$W_{\alpha,\beta}.\mathbb{R}^+) = \{f : f \text{ continuous on } \mathbb{R}^+, \sum_{k=-\infty}^{\infty} \sup_{2^k \le t < 2^{k+1}} t^\alpha |\ln t|^\beta |f.t)| < \infty\},$$

where $\alpha, \beta \in \mathbb{R}$ and $\widetilde{W}_{\alpha,\beta}.\mathbb{R}^+)$ is the class of locally Riemann integrable functions on $\mathbb{R}^+$ satisfying the same growth condition.

Corollary 4.4.

For $\alpha, \beta \ge 0$, let $f \in W_{\alpha,\beta}.\mathbb{R}^+)$ .or $\widetilde{W}_{\alpha,\beta}.\mathbb{R}^+)$ be positive and $\int_0^\infty f.t)t^{.\alpha-1/+i\xi}\, dt \ne 0$ for all $\xi \in \mathbb{R}$. Suppose μ is a positive regular Borel measure on $\{x : x \ge 0\}$ such that

$$\lim_{T\to\infty} \left(\frac{1}{T^\alpha |\ln T|^\beta} \int_0^\infty f\left(\frac{t}{T}\right) d\mu.t) - P.T) \right) = 0 \qquad .4.3)$$

for some bounded multiplicative periodic function of period a, that is $P.aT) = P.T)$. Then

$$\lim_{T\to\infty} \left(\frac{1}{T^\alpha |\ln T|^\beta} \int_0^\infty g\left(\frac{t}{T}\right) d\mu.t) - Q.T) \right) = 0$$

for all $g \in W_{\alpha,\beta}.\mathbb{R}^+)$.$\widetilde{W}_{\alpha,\beta}.\mathbb{R}^+)$, respectively), and $Q.aT) = Q.T)$ for all $T > 0$.

Proof. By using the transformation $x = \ln T$, $y = \ln t$, $\tilde{f}.y) = e^{-\alpha y} f.e^{-y})$, and $d\tilde{\mu}.y) = e^{\alpha y} d\mu.e^y)$, (4.3) is transformed into

$$\lim_{x\to\infty} \left(\frac{1}{x^\beta} \int_{-\infty}^\infty \tilde{f}.x - y)\, d\tilde{\mu}.y) - P.x) \right) = 0,$$

where $\tilde{f} \in W = \{h : h \text{ continuous on } \mathbb{R}, \sum_{k=-\infty}^\infty \sup_x |.h\chi_{[k,k+1/}).x)||k|^\beta < \infty\}$ and P is a bounded periodic function of period $\ln a$. Since $\tilde{\mu}$ satisfies $\overline{\lim}_{k\to-\infty} \tilde{\mu}[k, k+1) < \infty$, one applies Corollary 4.2 and the proof is complete. □

Corollary 4.5.

Suppose $\phi \ge 0$ on $\mathbb{R}^+$ and is integrable on $.0, h]$ for some $h > 0$. Let $f \in \widetilde{W}_{\alpha,\beta}.\mathbb{R}^+)$ with $\alpha \ge 0, \beta \ge 0$ be such that $\int_0^\infty f.t)t^{\alpha-1+i\xi}\, dt \ne 0$ for all ξ. Then

$$\lim_{T\to\infty} \left(\frac{1}{T^{\alpha-1}.\ln T)^\beta} \int_0^\infty \phi.Tt) f.t)\, dt \; - \; P.T) \right) = 0$$

if and only if

$$\lim_{T\to\infty} \left(\frac{1}{T^\alpha.\ln T)^\beta} \int_0^T \phi.t)\, dt \; - \; Q.T) \right) = 0$$

for some bounded multiplicative periodic functions P and Q.

Proof. By letting $d\mu.t) = \phi.t)dt$ and using a change of variables, the first expression reduces to

$$\lim_{T\to\infty} \left(\frac{1}{T^\alpha.\ln T)^\beta} \int_0^\infty \phi.t) f\left(\frac{t}{T}\right) dt \; - \; P.T) \right) = 0,$$

and the second expression reduces to

$$\lim_{T\to\infty} \left(\frac{1}{T^\alpha.\ln T)^\beta} \int_0^\infty \phi.t)\, \chi_{[0,1]}\left(\frac{t}{T}\right) dt \; - \; Q.T) \right) = 0.$$

Note that $g.t) = \chi_{[0,1]}$ is in $\widetilde{W}_{\alpha,\beta}.\mathbb{R}^+)$ for $\alpha \geq 0$ and $\int_0^\infty g.t)t^{\alpha-1+i\xi}\,dt = 1/.\alpha + i\xi) \neq 0$ for all ξ. Thus Corollary 4.4 can be applied. □

Note that Wiener's third Tauberian theorem is the special case when $\alpha, \beta = 0$ and $P.T)$ and $Q.T)$ are constants. Corollary 4.5 is used in [LMW] to estimate the Fourier transformation of the compactly supported L^2-solution of the two-scale dilation equation (as in (1.2)).

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