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Hausdorff Dimension of the Level Sets of Rademacher Series

by

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Summary. For $0 < \alpha < 1$, let $R(x) = \sum_{i=1}^{\infty} 2^{-\alpha i} R_i(x)$ for $0 \leqslant x < 1$, where $\{R_i\}_1^{\infty}$ is the sequence of Rademacher functions. If the distribution of R is absolutely continuous and the derivative is in L^p for some $p > 1$ then for [Leb] almost all y in the range of R, the y-level set of its graph has Hausdorff dimension $1 - \alpha$.

1. Introduction and preliminaries. Let $R_i(x)$, $i = 1, 2, \ldots$, denote the Rademacher functions on $\mathbb{R}$: $R_1(x)$ has period 1, and takes values 1 and -1 on the intervals $[0, 1/2)$ and $[1/2, 1)$ respectively, and $R_i(x) = R_1(2^{i-1}x)$ for $i > 1$. For $0 < \alpha < 1$, let

$$R(x) = R_\alpha(x) = \sum_{i=1}^{\infty} 2^{-\alpha i} R_i(x), \quad 0 \leqslant x < 1,$$

then the range of R is an interval $[-l, l]$, where $l = l_\alpha = \sum_{i=1}^{\infty} 2^{-\alpha i}$. Let

$$F(y) = F_\alpha(y) = |\{x \in [0,1) : R(x) < y\}|, \quad y \in \mathbb{R},$$

be the distribution function of R, where $|A|$ denotes the Lebesgue measure of a measurable subset A of $\mathbb{R}$. The following theorem is proved in [4]:

THEOREM 1.1. *If F is absolutely continuous and $F' \in L^p$ for some $p > 1$, then the Hausdorff dimension of the graph of R is $2 - \alpha$.*

The reader may refer to [4, 5, 7] for the motivation and the interest of the theorem.

In this note we will extend the theorem to conclude that under the same assumptions almost all the level sets $L(y) = \{(x, y) : R(x) = y\}$ of the graph are of Hausdorff dimension $1 - \alpha$ (Theorems 2.1, 2.3).

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Let $\mathcal{H}^s$ denote the s-Hausdorff measure. A general theorem related to the above is due to Marstrand [6, Thm III]: if A is a set with $0 < \mathcal{H}^s(A) < \infty$, where $s > 1$, then for almost all points x of A, the intersection of A with almost every line passing through x has dimension $s-1$ and has finite $(s-1)$-dimensional measure. This result together with Theorem 1.1, however, is not enough to derive Theorems 2.1, 2.3.

Throughout the letter n will denote a positive integer, $\mathcal{I}$ will be the family of all dyadic intervals of $[0,1)$, and $\mathcal{I}_n$ will be the subfamily of $\mathcal{I}$ consisting of those members of size 2^{-n}.

For any $I \in \mathcal{I}$, let $F_I(y) = |\{x \in I : R(x) < y\}|$. It is easy to prove that if F is absolutely continuous, than F_I is absolutely continuous, and $F'(y) \geqslant F_I'(y)$. The following self-similarity property is the key of our consideration, its proof can be found in [4, Lemma 2.2]:

For any $I \in \mathcal{I}_n$, let $b_I = \sum_{i=1}^n 2^{-\alpha i} R_i(x)$, $x \in I$, then for each $y \in [-l,l]$ we have

$$F_I(y) = 2^{-n} F(2^{\alpha n}(y - b_I)).$$

(See Fig. 1, the graph inside the rectangle is affinely similar to the whole graph, and the distribution F_I is related to F as indicated). The statement implies that

$$F_I'(y) = 2^{(\alpha-1)n} F'(2^{\alpha n}(y - b_I)), \tag{1.1}$$

or, equivalently,

$$F_I'(y) = |I|^{1-\alpha} F'(|I|^{-\alpha}(y - b_I)). \tag{1.2}$$

In the following we will use (1.1) or (1.2) whichever is convenient. Let $R(I)$, $I \in \mathcal{I}$, be the image of I under R, then $R(I)$ is an interval and $|R(I)| = 2l|I|^\alpha$.

PROPOSITION 1.1. *Let $1 \leqslant p \leqslant \infty$. If F is absolutely continuous and F' restricted to J is in $L^p(J)$ for some interval J in the range of R, then $F' \in L^p$.*

Proof. Assume $1 \leqslant p < \infty$, the case $p = \infty$ can be proved similarly. Choose n and $I \in \mathcal{I}_n$ so that $R(I) = [-l \cdot 2^{-\alpha n}, l \cdot 2^{-\alpha n}] \subseteq J$. By a change of variables and using (1.1), we have

$$\int_{-l}^{l} (F'(y))^p\, dy = 2^{\alpha n} \int_{R(I)} (F'(2^{\alpha n}(x - b_I)))^p\, dx$$

$$= 2^{(\alpha+p-\alpha p)n} \int_{R(I)} (F_I'(x))^p\, dx \leqslant 2^{(\alpha+p-\alpha p)n} \int_J (F'(x))^p\, dx < \infty,$$

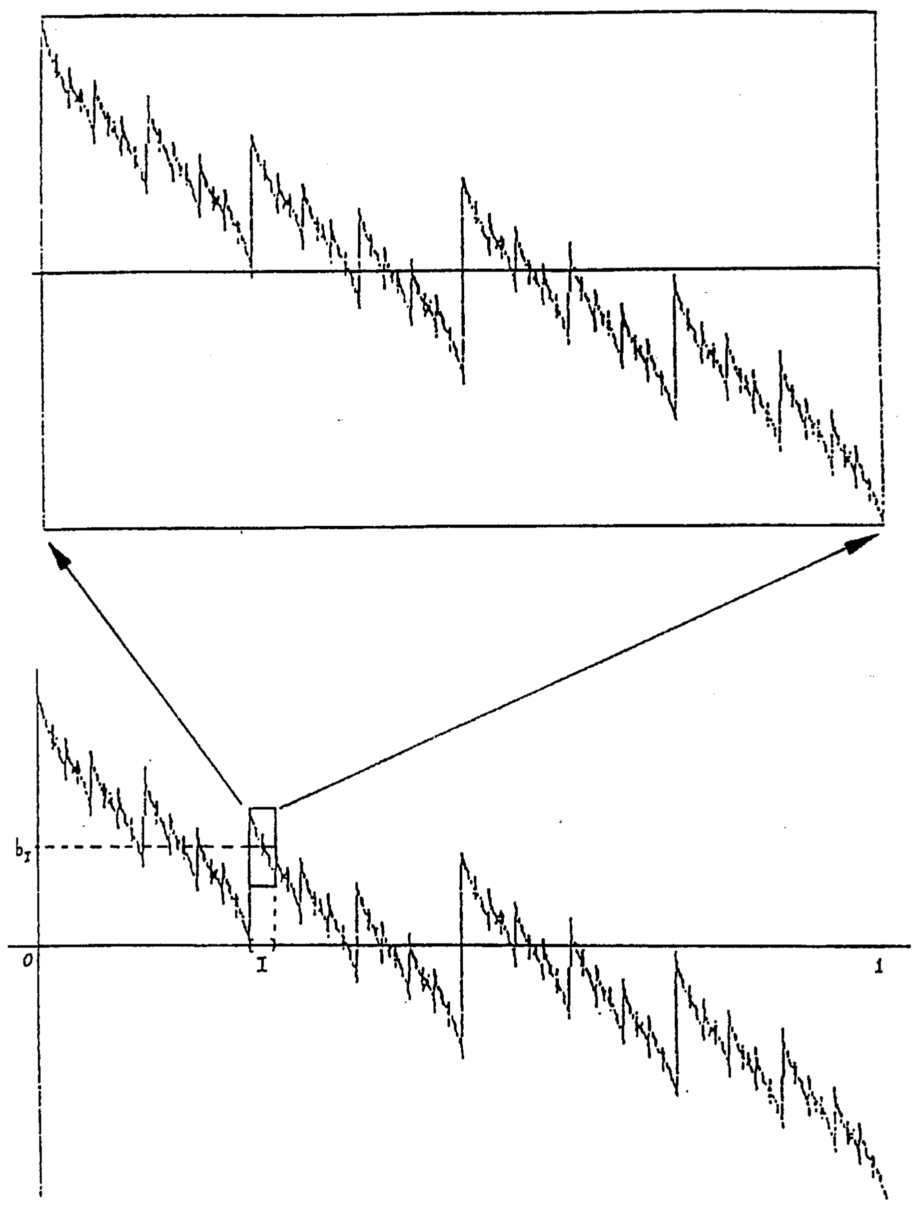

Fig. 1.

and the proposition follows.

PROPOSITION 1.2. *Let* $E = \{y \in [-l, l] : F'(y) = 0\}$, *then either* $|E| = 0$ *or* $|E| = 2l$.

Proof. Suppose that $|E| > 0$. We will show that $|E| > \beta$ for any $0 < \beta < 2l$. Since for any $I \in \mathcal{I}$, $R(I)$ is an interval, we can choose $I \in \mathcal{I}_n$ for n large enough such that

$$|R(I) \cap E|/|R(I)| > (2l)^{-1}\beta. \tag{1.3}$$

Observe that for every $y \in R(I) \cap E$, $F_I'(y) = 0$, hence $F'(2^{\alpha n}(y - b_I)) = 0$ by (1.1). It follows that

$$\{2^{\alpha n}(y - b_I) : y \in R(I) \cap E\} \subseteq E.$$

By (1.3) we obtain

$$|E| \geqslant 2^{\alpha n}|R(I) \cap E| > 2^{\alpha n}|R(I)|(2l)^{-1}\beta = \beta.$$

2. The level sets. For any $y \in [-l, l]$, let $L(y)$ be the level set of the graph of R on the y-level, let $I_y = I \times \{y\}$, where $I \in \mathcal{I}$, and let

$$\mathcal{I}_n(y) = \{I_y : I \in \mathcal{I}_n,\ I_y \cap L(y) \neq \emptyset\},$$

then $\mathcal{I}_n(y)$ is a cover of $L(y)$ and is unique among the "dyadic intervals" of equal size 2^{-n}.

THEOREM 2.1. *If* F *is absolutely continuous with* $F' \in L^\infty$, *then* $F'(y) > 0$ *for almost all* y *in* $[-l, l]$. *For such* y, $0 < \mathcal{H}^{1-\alpha}(L(y)) < \infty$, *and hence* $\dim L(y) = 1 - \alpha$.

Proof. It follows from Proposition 1.2 that $0 < F'(y) < M$ holds for almost all y in $[-l, l]$. Let $y \in [-l, l]$, we first show that $\mathcal{H}^{1-\alpha}(L(y)) < \infty$. Since $R(I)$, $I \in \mathcal{I}_n$, is an interval of length $2l \cdot 2^{-\alpha n}$,

$$\begin{aligned} 2^{-n}\ \#\mathcal{I}_n(y) &= \Big|\bigcup\{I \in \mathcal{I}_n : I_y \cap L(y) \neq \emptyset\}\Big| = \Big|\bigcup\{I \in \mathcal{I}_n : y \in R(I)\}\Big| \\ &\leqslant |\{x \in [0,1) : |R(x) - y| < 2l \cdot 2^{-\alpha n}\}| \\ &= F(y + 2l \cdot 2^{-\alpha n}) - F(y - 2l \cdot 2^{-\alpha n}). \end{aligned}$$

Now, considering $\mathcal{I}_n(y)$ as a cover of $L(y)$, we have

$$\begin{aligned} \sum_{I_y \in \mathcal{I}_n(y)} |I_y|^{1-\alpha} &= \#\mathcal{I}_n(y) \cdot 2^{-n(1-\alpha)} \\ &\leqslant [F(y + 2l \cdot 2^{-\alpha n}) - F(y - 2l \cdot 2^{-\alpha n})]/2^{-\alpha n}, \end{aligned}$$

which converges to $4lF'(y)$ as $n \to \infty$, therefore $\mathcal{H}^{1-\alpha}(L(y)) < \infty$.

To prove $\mathcal{H}^{1-\alpha}(L(y)) > 0$, let $y \in [-l, l]$ with $F'(y) > 0$ and let $\mathcal{C} = \{I_y : I \in \mathcal{I}\}$ be an arbitrary cover of $L(y)$. By observing that $\sum_{I_y \in \mathcal{C}} F_I'(y) = F'(y)$ and using (1.2) we have

$$\begin{aligned}(2.1)\qquad \sum_{I_y \in C} |I_y|^{1-\alpha} &= \sum_{I_y \in C} F_I'(y)/F'(|I|^{-\alpha}(y - b_I)) \\ &\geqslant M^{-1} \sum_{I_y \in C} F_I'(y) = M^{-1} F'(y) > 0.\end{aligned}$$

Hence $\mathcal{H}^{1-\alpha}(L(y)) > 0$. This completes the proof of the theorem.

Erdös [2] showed that for any positive integer k, there is an $\alpha_k > 0$ (sufficiently close to 0) so that for almost all $0 < \alpha \leqslant \alpha_k$, F_α has derivative of order k. Propositions 1.1 and 1.2 imply that for such an α, $F_\alpha' \in L^\infty$ and the set of those y so that $F_\alpha'(y) = 0$ is a closed nowhere dense subset of $[-l_\alpha, l_\alpha]$, hence $0 < \mathcal{H}^{1-\alpha}(L(y)) < \infty$ holds for every y in an open dense subset of Lebesgue measure $2l_\alpha$ in $[-l_\alpha, l_\alpha]$.

For the special case where $\alpha = 1/k$, $k = 1, 2, \ldots$, it is known that $F_{1/k}$ has derivative of order k, actually its derivative is positive for every $y \in (-l_{1/k}, l_{1/k})$. Theorem 2.1 hence improves the corresponding result of Beyer [1]: For $k = 1, 2, \ldots$, $\dim L(y) = 1 - \alpha$ for *every* $y \in (-l_{1/k}, l_{1/k})$. A simple proof for $0 < F_{1/k}'(y) < M < \infty$ for *every* $y \in (-l_{1/k}, l_{1/k})$ is as follows: Let $\beta = 2^{-\alpha}$, since the characteristic function of F_α is

$$\varphi(u, \beta) = \prod_{i=1}^{\infty} \cos \beta^i u.$$

A direct computation shows that for any $k \geqslant 1$,

$$\varphi(u, \beta) = \varphi(u, \beta^k)\varphi(u\beta^{-1}, \beta^k) \ldots \varphi(u\beta^{-(k-1)}, \beta^k).$$

That $\varphi(u, 2^{-1}) = \sin u/u$ implies that for $k \geqslant 1$

$$\varphi(u, 2^{-1/k}) = \prod_{i=0}^{k-1} \sin(2^{i/k}u)/(2^{i/k}u).$$

For $i = 0, \ldots, k-1$, since $\sin(2^{i/k}u)/(2^{i/k}u)$ is the characteristic function of the uniform distribution G_i over the interval $(-2^{i/k}, 2^{i/k})$, $F_{1/k}$ equals the convolution $G_0 * G_1 * \ldots * G_{k-1}$; it is hence supported by $(-\sum_{i=0}^{k-1} 2^{i/k}, \sum_{i=0}^{k-1} 2^{i/k}) = (-l_{1/k}, l_{1/k})$ and has a positive, bounded derivative at every point in the interval $(-l_{1/k}, l_{1/k})$.

In the following we will consider the case $F' \in L^p$ for some $p > 1$. In the proof of Theorem 2.1, we use the boundedness of F' to control the inequality in (2.1). We will first show that this inequality is still valid for almost all $y \in [-l, l]$ in the present case.

Let $\delta > 0$ be an arbitrary fixed number. For any $I \in \mathcal{I}$, let

$$\mathbf{E}_I = \{y \in R(I) : [F'(|I|^{-\alpha}(y - b_I))]^{-1} < |I|^\delta\}.$$

For any fixed $y \in [-l, l]$ and for any cover $\mathcal{C}(y)$ of $L(y)$ of the form $\{I_y = I \times \{y\} : I \in \mathcal{I}\}$, let

$$\mathcal{C}'(y) = \{I_y \in \mathcal{C}(y) : y \in \mathbf{E}_I\}$$

be the "exceptional" intervals in the cover. Let

$$\tau(y) = \lim_n \sup_{|\mathcal{C}(y)| < 2^{-n}} \sum_{I_y \in \mathcal{C}'(y)} F_I'(y)$$

where $|\mathcal{C}(y)| = \sup\{|I_y| : I_y \in \mathcal{C}(y)\}$ (If $\mathcal{C}'(y) = \emptyset$, define $\sum_{I_y \in \mathcal{C}'(y)} F_I'(y) = 0$). Then

$$\tau(y) \leqslant \lim_n \sum_{i=n}^{\infty} \sum_{I \in \mathcal{I}_i} F_I'(y) \cdot \chi_{\mathbf{E}_I}(y).$$

LEMMA 2.2. *Let* $A = \{y \in [-l, l] : \tau(y) \geqslant F'(y)/2\}$. *If* F *is absolutely continuous and* $F' \in L^p$ *for some* $p > 1$, *then* $|A| = 0$.

Proof. Let ν be the probability measure defined by $\nu((-\infty, y]) = F(y)$, then

$$\nu(A) = \int_A F'(y)\,dy \leqslant 2\int_A \tau(y)\,dy \leqslant 2 \lim_n \sum_{i=n}^{\infty} \sum_{I \in \mathcal{I}_i} \int_{\mathbf{E}_I} F_I'(y)\,dy.$$

Applying (1.1) and following by a change of variables we have for $I \in \mathcal{I}_i$

$$\int_{\mathbf{E}_I} F_I'(y)\,dy = 2^{(\alpha-1)i} \int_{\mathbf{E}_I} F'(2^{\alpha i}(y - b_I))\,dy = 2^{-i} \int_{E_i(\delta)} F'(y)\,dy$$

where $E_i(\delta) = \{y \in [-l, l] : F'(y) \geqslant 2^{i\delta}\}$. Note that $|E_i(\delta)| \leqslant 2^{-i\delta}$ and $F' \in L^p$ for some $p > 1$, the Hölder inequality yields

$$\int_{E_i(\delta)} F'(y)\,dy \leqslant |E_i(\delta)|^{1/p'} ||F||_p = C \cdot 2^{-i\delta/p'},$$

where $1/p + 1/p' = 1$. Therefore

$$\nu(A) \leqslant 2 \lim_n \sum_{i=n}^{\infty} \int_{E_i(\delta)} F'(y)\,dy = 0.$$

Hence $|A| = 0$ by Proposition 1.2 and $\nu(A) = \int_A F'(y)dy$.

THEOREM 2.3. *If* F *is absolutely continuous and* $F' \in L^p$ *for some* $p > 1$, *then* $\dim L(y) = 1 - \alpha$ *for almost all* $y \in [-l, l]$.

Proof. Let $B = \{y \in [-l,l] : F'(y) = 0 \text{ or does not exist}\}$, then $|B| = 0$ by Proposition 1.2. Let $D = A \cup B$, where A is as in Lemma 2.2, then $|D| = 0$. We will show that if $y \notin D$ then $\dim L(y) = 1 - \alpha$. That $\dim L(y) \leqslant 1 - \alpha$ follows from the same proof as in Theorem 2.1, we only need to prove the reverse inequality.

Let $y \notin D$. Since $y \notin A$, by Lemma 2.2, $\tau(y) < F'(y)/2$. By the definition of τ, there is an integer k so that for all $n > k$ and for any cover $\mathcal{C}(y) = \{I_y : I \in \mathcal{I}\}$ of $L(y)$ with $|\mathcal{C}(y)| \leqslant 2^{-n}$,

$$\sum_{I_y \in \mathcal{C}'(y)} F_I'(y) < F'(y)/2. \tag{2.2}$$

Note that $\sum_{I_y \in \mathcal{C}(y)} F_I'(y) = F'(y)$, thus (2.2) is equivalent to

$$\sum_{I_y \in \mathcal{C}(y) \backslash \mathcal{C}'(y)} F_I'(y) \geqslant F'(y)/2.$$

Let $\mathcal{C}(y) = \{I_y : I \in \mathcal{I}\}$ be any cover of $L(y)$ with $|\mathcal{C}(y)| \leqslant 2^{-n}$, where $n > k$. By the definition of $\mathcal{C}'(y)$, $I_y \in \mathcal{C}(y) \backslash \mathcal{C}'(y)$ implies that $[F'(|I|^{-\alpha}(y - b_I))]^{-1} \geqslant |I|^{\delta}$, combining this with (1.2) we have

$$\begin{aligned}\sum_{I_y \in \mathcal{C}(y)} |I_y|^{1-\alpha-\delta} &= \sum_{I_y \in \mathcal{C}(y)} F_I'(y)[F'(|I|^{-\alpha}(y - b_I))]^{-1}|I|^{-\delta} \\ &\geqslant \sum_{I_y \in \mathcal{C}(y) \backslash \mathcal{C}'(y)} F_I'(y)[F'(|I|^{-\alpha}(y - b_I))]^{-1}|I|^{-\delta} \\ &\geqslant \sum_{I_y \in \mathcal{C}(y) \backslash \mathcal{C}'(y)} F_I'(y) \geqslant F'(y)/2 > 0.\end{aligned}$$

Since $\delta > 0$ is arbitrary, $\dim(L(y)) \geqslant 1 - \alpha$, proving the theorem.

We remark that Theorem 1.1 is a corollary of Theorem 2.3 by using a theorem of Marstrand [3, Thm 5.8].

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